Physical Yukawa Couplings in Heterotic String Theory from Localisation

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work in collaboration with
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3. stabilising the moduli and inserting their values into the moduli-dependent expressions for the physical Yukawa couplings to obtain actual numerical values (hard)

Context:

- Heterotic $E_8 \times E_8$ string theory on Calabi-Yau threefold X
- Observable vector bundle $V \to X$ with structure group $H \subset E_8$
- Low-energy gauge group $G = \mathcal{C}_{E_8}(H)$

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- In this talk, V is a sum of line bundles

Holomorphic Yukawa Couplings

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Holomorphic Yukawa couplings are independent of representatives

$$\int_X \Omega \wedge \left(\nu_i + \bar{\partial}\xi_i\right) \wedge \left(\nu_j + \bar{\partial}\xi_j\right) \wedge \left(\nu_k + \bar{\partial}\xi_k\right) = \int_X \Omega \wedge \nu_i \wedge \nu_j \wedge \nu_k \ ,$$

The Matter Field Kähler Metric

$$G_{ij}C^i\bar{C}^j \subset K$$

$$\begin{aligned} G_{ij} &= \frac{1}{2\mathcal{V}}(\nu_i, \nu_j) \\ &= \frac{1}{2\mathcal{V}}\int_X \nu_i \wedge \bar{*}_V(\nu_j) = -\frac{i}{4\mathcal{V}}\int_X \nu_i \wedge J \wedge J \wedge (H\bar{\nu}_j) \end{aligned}$$

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This is too hard.. What can we do?

Let's look at something simpler, line bundles on $\mathbb{P}^1$.

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We want to find harmonic, $\mathcal{L}$-valued forms $\hat{\nu}$ on $\mathbb{P}^1$. These must be globally well-defined, hence by demanding the correct transformation property between the two patches of $\mathbb{P}^1$ and imposing $\bar{\partial}\hat{\nu} = 0$ and $\partial(\hat{H}\hat{\nu}) = 0$ we obtain

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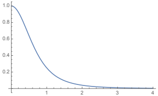
where $P_{-k-2}(\bar{z})$ is a polynomial of degree $-k-2$ in $\bar{z}$. Then

$$(\hat{\nu}, \hat{\nu}) = \int_{\mathbb{P}^1} \hat{\nu}\hat{H}\bar{\hat{\nu}} = \int_{\mathbb{P}^1} |P|^2 \kappa^k dz\, d\bar{z} \quad \text{localises for large } |k|$$

Plot the integrand $|P|^2\kappa^k$

	$P = 1$	$P = \bar{z}$	$P = \bar{z}^2$	$P = \bar{z}^3$
$k = -2$				

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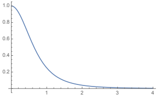
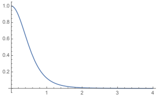
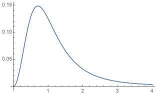
$k = -2$				
$k = -3$				

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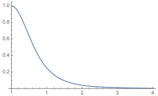
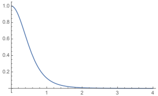
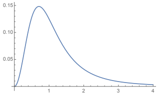
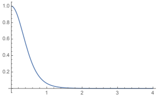
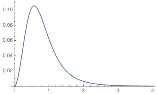
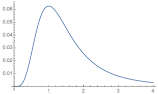
$k = -2$				
$k = -3$				
$k = -4$				

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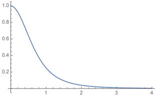

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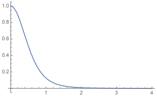

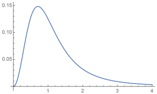

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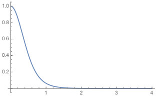
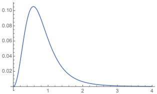
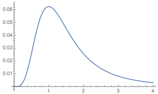
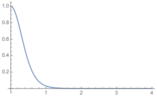
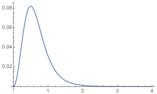


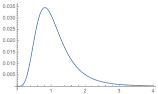


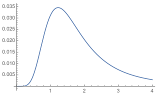


$k = -5$









An E_6 Model on the Tetraquadric

Consider a generic tetra-quadric hypersurface
$X = \{p = 0\} \subset \mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; affine coordinates z_1, z_2, z_3, z_4.

$$X \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}2\\2\\2\\2\end{bmatrix}^{4,68} \qquad V = L_1 \oplus L_2 \oplus L_3 \;=\; \begin{bmatrix}-2 & 0 & 2\\ 0 & -2 & 2\\ 1 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$$

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The low-energy gauge group is $E_6 \times S(U(1)^3)$. The $\mathbf{27}$-multiplets carry $U(1)$ charges.

Cohomology of L_i via Koszul sequence $0 \to \mathcal{N}^* \otimes \mathcal{L} \to \mathcal{L} \to L \to 0$ where $\mathcal{N} = \mathcal{O}_{\mathcal{A}}(2,2,2,2)$ and $L = \mathcal{L}|_{\mathcal{A}}$

$$H^1(X, \mathcal{L}_1) \simeq H^1(\mathcal{A}, \mathcal{L}_1) \simeq \mathbb{C}^2$$
$$H^1(X, \mathcal{L}_2) \simeq H^1(\mathcal{A}, \mathcal{L}_2) \simeq \mathbb{C}^2$$
$$H^1(X, \mathcal{L}_3) \simeq H^1(\mathcal{A}, \mathcal{L}_3) \oplus H^2(\mathcal{A}, \mathcal{L}_3 \otimes \mathcal{N}^*) \simeq \mathbb{C}^3 \oplus \mathbb{C}^9$$

Holomorphic Yukawa Coupling

The only non-trivial Yukawa coupling $\lambda(\nu_1, \nu_2, \nu_3)$ corresponds to

- $\nu_1 = \hat{\nu}_1|_X \ , \qquad \hat{\nu}_1 \in H^1(\mathcal{A}, \mathcal{L}_1)$
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- $\nu_3 = \hat{\nu}_3|_X \ , \qquad \bar{\partial}\hat{\nu}_3 = p\,\hat{\omega} \ , \qquad \hat{\omega} \in H^2(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{L}_3)$

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We can write the ambient space forms explicitly,

$$\hat{\nu}_1 = \frac{1}{\kappa_1^2}(a_1 + b_1 z_3) d\bar{z}_1 \ , \qquad \hat{\nu}_2 = \frac{1}{\kappa_2^2}(a_2 + b_2 z_3) d\bar{z}_2$$

$$\hat{\omega} = \frac{1}{\kappa_3^4 \kappa_4^2}(a_3 + b_3 \bar{z}_3 + c_3 \bar{z}_3^2) d\bar{z}_3 \wedge d\bar{z}_4$$

$$\lambda(\nu_1, \nu_2, \nu_3) = \int_X \Omega \wedge \nu_1 \wedge \nu_2 \wedge \nu_3 = \frac{1}{\pi} \int_{\mathbb{C}^4} d^4 z \wedge \hat{\nu}_1 \wedge \hat{\nu}_2 \wedge \hat{\omega}_3$$

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$$\boxed{\lambda(\nu_1, \nu_2, \nu_3) = \frac{(2\pi)^3}{3} (2\, a_1\, a_2\, a_3 + 2\, b_1\, b_2\, c_3 + a_1\, b_2\, b_3 + b_1\, a_2\, b_3)}$$

Let us restrict to the three multiplets that correspond to

$$\hat{\nu}_1 = \frac{1}{\kappa_1^2} d\bar{z}_1 \ , \qquad \hat{\nu}_2 = \frac{1}{\kappa_2^2} d\bar{z}_2$$
$$\hat{\omega} = \frac{1}{\kappa_3^4 \kappa_4^2} d\bar{z}_3 \wedge d\bar{z}_4$$

The holomorphic Yukawa coupling takes the value $\dfrac{16\pi^3}{3}$.

Next, we need to compute the normalisation of the forms $\hat{\nu}_1$, $\hat{\nu}_2$ and $\hat{\nu}_3$ defined by $\bar{\partial}\hat{\nu}_3 = p\,\hat{\omega}$.

Normalisation integrals

The normalisation integrals for the above forms localise around the origin $z_1 = z_2 = z_3 = z_4 = 0$. By a suitable coordinate redefinition on the embedding projective spaces, the origin can be chosen to be a point on X.

The normalisation integrals have to be carried out on X, not on $\mathcal{A}$. I'm skipping a long technical discussion on how the restrictions to X of the above ambient space harmonic forms are related to forms on X that are harmonic with respect to the Ricci-flat metric.

Physical Yukawa coupling

$$\frac{1}{2\mathcal{V}}(\hat{\nu}_1, \hat{\nu}_1) \approx \frac{\pi}{4t_1}$$
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With these normalisations, the above holomorphic Yukawa coupling translates into the following physical Yukawa coupling

$$\boxed{Y(C_1, C_2, C_3) \approx \frac{4^5 \pi^{3/2}}{3} t_1 t_2 \sqrt{\frac{t_3}{5t_1t_2 + t_1t_3 + t_2t_3}}}$$

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Thank you for listening!