Anomalies, string universality, and model building

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Based on arXiv:1710.04218
with H. Hayashi, K. Ohmori, Y. Tachikawa and K. Yonekura

and work to appear
with M. Montero.

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I will try to explain how to formulate these questions precisely, and partially answer the last question. (Miguel will answer the rest of the questions.)

Why these questions

The common thread in all these topics is that we are trying to study whether the theory makes sense on arbitrary manifolds. Recent developments [Dai, Freed '94], [Witten '15] have shed new light on this old topic.

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In fact, to my knowledge the constraints I present are best motivated when thinking about quantum gravity: we expect that quantum gravity fluctuations can freely change the topology of spacetime, so imposing consistency of a quantum theory on manifolds of arbitrary topology seems very natural!

Review of anomalies (I)

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A_G for G, and compute the path integral

$$Z(A_G) = \int [D\psi] e^{-S(A_G,\psi)} \tag{1}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

Denote by $\mathcal{M}$ the space of all A_G. We have an anomaly whenever $Z(A_G)$ is not well defined as a function on the manifold $\mathcal{M}/G$:

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

A note on terminology

"Global anomalies" is also sometimes used in the literature to mean "local anomalies (2.) of global symmetries (1.)".

Unfortunately, in the literature the word "global" is used in three different ways:

1. Local (gauge) vs. global symmetry.
2. Local vs. global anomalies (statement about connection space; perturbative vs. non-perturbative computations).
3. Local vs. global features on spacetime. (I.e. whether the topology of spacetime matters.)

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I will be discussing global anomalies (2.) for local (gauge) symmetries (1.) that may or may not depend on the topology of spacetime (3.).

Review of anomalies (II)

In general, $Z(A_G)$ is a section of some bundle over $\mathcal{M}/G$. If the bundle is non-trivial the theory is still consistent; we say that we have a 't Hooft anomaly, which may be local or global.

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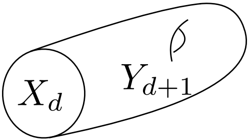
We will consider the case in which there are no local anomalies. How do we detect a possible global anomaly?

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X_d is the boundary of some manifold Y_{d+1}, over which all the relevant structures on X_d extend.



We define the path integral of a fermion ψ on X_d as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_{Y_{d+1}})} \tag{2}$$

with

$$\eta(\mathcal{D}_{Y_{d+1}}) = \frac{\dim\ker \mathcal{D}_{Y_{d+1}} + \sum_{\lambda\neq 0} \operatorname{sign}(\lambda)}{2}\,. \tag{3}$$

[*] For the experts, this is the same η that appears in the APS index theorem. More on this soon.

Why is this prescription useful

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Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

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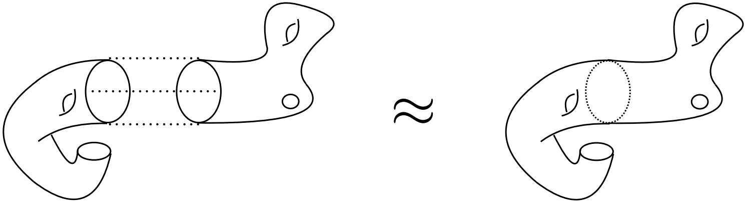
and it is "local", in the sense that η behaves nicely under gluing:

$$e^{2\pi i \eta(\mathcal{D}_A)} e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})} \tag{5}$$



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Anomalies, in this language, come from situations in which the phase of the partition function depends on the choice of Y_{d+1}:

$$e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})} \neq e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} \tag{6}$$

even if $\partial Y_{d+1} = \partial Y'_{d+1} = X_d$.

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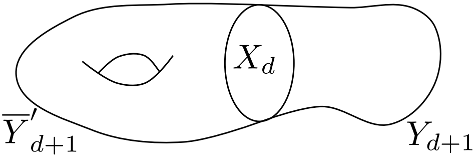
Gluing Y_{d+1} and $\overline{Y}_{d+1}$ over X_d to form the closed manifold W_{d+1}, we find that the partition function is well defined as a function of the fields on X_d only if on every such W_{d+1}

$$e^{-2\pi i\,\eta(\mathcal{D}_{W_{d+1}})} = e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})}/e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} = 1 \tag{7}$$



The APS index theorem

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$$\operatorname{ind}(\not{D}_{Z_{d+2}}) = \eta(D_{W_{d+1}}) + \int_{Z_{d+2}} \hat{A}(R)\,\mathrm{ch}(F)\,. \qquad (8)$$

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Since the index is an integer, this leads to

$$\exp(-2\pi i\,\eta(\mathcal{D}_{W_{d+1}})) = \exp\left(2\pi i \int_{Z_{d+2}} \hat{A}(R)\,\mathrm{ch}(F)\right)\,. \qquad (9)$$

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The expression on the right hand side is the local anomaly polynomial, so in the absence of local anomalies (easily checked, I'll assume it from now on) we have that

$$\exp(2\pi i\,\eta(\mathcal{D}_{W_{d+1}})) = 1 \tag{10}$$

whenever W_{d+1} is a boundary.

Anomalies and bordism

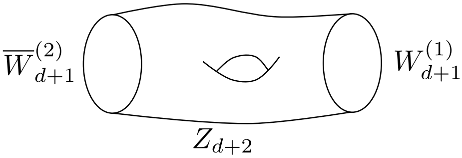
What this means is that if we have some manifold Z_{d+2} such that

$$\partial Z_{d+2} = W^{(1)}_{d+1} - W^{(2)}_{d+1} \tag{11}$$

then

$$\exp(2\pi i\,\eta(\mathcal{D}_{W^{(1)}_{d+1}})) = \exp(2\pi i\,\eta(\mathcal{D}_{W^{(2)}_{d+1}})) \tag{12}$$

This is a huge simplification! For the purposes of anomalies any two manifolds which can be connected via a third manifold are then equivalent: $W^{(1)}_{d+1} \sim W^{(2)}_{d+1}$



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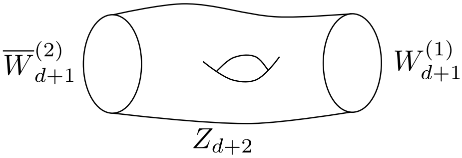
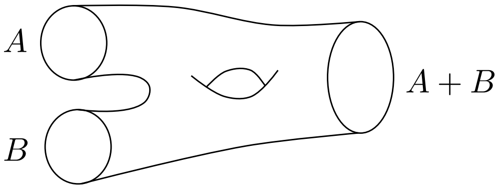


This equivalence relation is known as **bordism**, and the resulting equivalence class of manifolds is denoted Ω_{d+1}.

Some basic properties of bordism and η

The equivalence class Ω_{d+1} is an abelian group, under disjoint union of manifolds:



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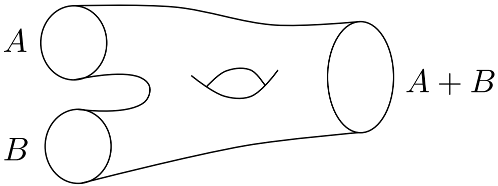

We have that

$$e^{2\pi i\eta(\mathcal{D}_A)}e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})} \tag{13}$$

so the global anomaly is a *homomorphism*

$$e^{e\pi i\eta}\colon \Omega_{d+1} \to U(1) \tag{14}$$

So, for example, if $\Omega_{d+1} = 0$, the anomaly necessarily vanishes.

Decorating bordism

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We are interested in gauge theories. That is, in understanding the partition function as a function of a principal bundle E_G on the manifold, for some group G. In this formalism this is encoded in decorating the manifolds with maps $W_{d+1} \to BG$, with BG the "classifying space of G". Some examples

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$\mathbb{Z}_2$	$\mathbb{RP}^\infty$
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In general, bordism groups of Spin manifolds W_{d+1} decorated with a map to $\mathcal{M}$ are denoted by

$$\Omega_{d+1}^{\mathrm{Spin}}(\mathcal{M})\,. \tag{15}$$

The strategy

The beauty of the Dai-Freed approach is that we can formulate necessary and sufficient conditions for quantum consistency on any manifold X_d for a theory with group G:

- Construct all the bordism groups in one dimension higher with the right structure. For instance $\Omega_{d+1}^{\mathrm{Spin}}(BG)$.
- The theory is anomaly free iff the $e^{2\pi i\,\eta}$ homomorphism gives 1 for every equivalence class in $\Omega_{d+1}^{\mathrm{Spin}}(BG)$.

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Otherwise, we need to find some generators of $\Omega_{d+1}^{\mathrm{Spin}}(BG)$ on which we can compute η. Not an easy task!

How to compute $\Omega_*^{\mathrm{Spin}}(BG)$

In many useful ways, one can think of $\Omega_*^{\mathrm{Spin}}(\mathcal{M})$ as a "generalized homology theory"

$$\Omega_*^{\mathrm{Spin}}(\mathcal{M}) \cong \mathcal{H}_*(\mathcal{M})\,. \tag{16}$$

This $\mathcal{H}$ behaves like ordinary homology, except for:

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If we have a fibration $0 \to F \to E \to B \to 0$, then for an ordinary homology we can "assemble" $H_*(E)$ starting from

$$\sum_{p+q=k} H_p(B, H_q(F)) \Rightarrow H_k(E) \tag{18}$$

This is known as a **spectral sequence**.

How to compute $\Omega_*^{\mathrm{Spin}}(BG)$

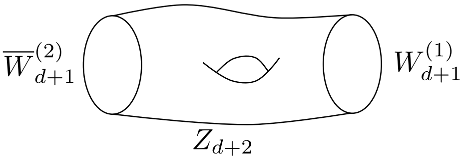
A similar idea works for any generalized homology (this is known as the **Atiyah-Hirzebruch spectral sequence**)

$$\sum_{p+q=k} H_p(B, \mathcal{H}_q(F)) \Rightarrow \mathcal{H}_k(E) \tag{19}$$

Now, any space X fits in a fibration $0 \to \mathrm{pt} \to X \to X \to 0$, so

$$\sum_{p+q=k} H_p(X, \mathcal{H}_q(\mathrm{pt})) \Rightarrow \mathcal{H}_k(X) \tag{20}$$

For us $X = BG$ and $\mathcal{H} = \Omega^{\mathrm{Spin}}$ so this gives a way of seeing how adding bundles modifies ordinary bordism $\Omega_k^{\mathrm{Spin}} \cong \Omega_k^{\mathrm{Spin}}(\mathrm{pt})$



Result of the computation for some Lie groups

$\mathbf{G}$	$\Omega^{\mathrm{Spin}}_{\mathbf{k}}(\mathbf{BG})$								
	0	1	2	3	4	5	6	7	8
$SU(2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$4\mathbb{Z}$
$SU(n>2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
$USp(2k>2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$5\mathbb{Z}$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbf{0}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbf{0}$	–	–	–
$Spin(n \geq 8)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
$SO(n \geq 3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z}_2, \mathbb{Z}_2)$	$\mathbf{0}$	$e(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2)$	$\mathbf{0}$	–	–	–
E_6, E_7, E_8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$2\mathbb{Z}$
G_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
F_4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	–

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The global form matters: $\Omega_k^{\mathrm{Spin}}(BSO(n)) \neq \Omega_k^{\mathrm{Spin}}(BSpin(n))$.

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$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbf{0}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbf{0}$	–	–	–
$Spin(n \geq 8)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
$SO(n \geq 3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z}_2, \mathbb{Z}_2)$	$\mathbf{0}$	$e(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2)$	$\mathbf{0}$	–	–	–
E_6, E_7, E_8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$2\mathbb{Z}$
G_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
F_4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	–

The global form matters: $\Omega_k^{\text{Spin}}(BSO(n)) \neq \Omega_k^{\text{Spin}}(BSpin(n))$.

For $d = 4$ we have $\Omega_{d+1}^{\text{Spin}}(BG) = 0$ for all cases we checked, except the symplectic groups (which have Witten's $\mathbb{Z}_2$ anomaly [Witten '82]).

Result of the computation for some Lie groups

$\mathbf{G}$	$\boldsymbol{\Omega}_{\mathbf{k}}^{\mathrm{Spin}}(\mathbf{BG})$								
	0	1	2	3	4	5	6	7	8
$SU(2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$4\mathbb{Z}$
$SU(n>2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
$USp(2k>2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$5\mathbb{Z}$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbf{0}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbf{0}$	–	–	–
$Spin(n \geq 8)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
$SO(n \geq 3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z}_2, \mathbb{Z}_2)$	$\mathbf{0}$	$e(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2)$	$\mathbf{0}$	–	–	–
E_6, E_7, E_8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$2\mathbb{Z}$
G_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$2\mathbb{Z}$	$\mathbf{0}$	–	–	–
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The global form matters: $\Omega_k^{\mathrm{Spin}}(BSO(n)) \neq \Omega_k^{\mathrm{Spin}}(BSpin(n))$.

For $d=4$ we have $\Omega_{d+1}^{\mathrm{Spin}}(BG)=0$ for all cases we checked, except the symplectic groups (which have Witten's $\mathbb{Z}_2$ anomaly [Witten '82]).

The computation gets harder as we increase the dimension: at high dimensions one cannot always read the answer in this way.

The 8d string swampland

One original motivation for looking at global anomalies is the 8d/9d swampland [Vafa '05]: in $d \in \{9, 8, 7\}$ supersymmetry implies that the fermions are in the adjoint representation, which is real, so these theories have no local anomalies.

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How much of this gap is anomalies, and how much subtle quantum gravity effects?

String compactifications down to 9d

There are four known components of the $\mathcal{N} = 1$ moduli space one can construct this way (for a detailed analysis see [Aharony, Komargodski, Patir '07])

- **Rank 2 (a):**
  - M-theory on the Klein bottle.
- **Rank 2 (b):**
  - IIA with $O8^+$ and $O8^-$.
- **Rank 10:**
  - M-theory on Möbius band.
  - CHL string. [Chaudhury, Hockney, Lykken '95]
- **Rank 18:**
  - M-theory on the cylinder.
  - Heterotic on S^1.
  - IIA with two $O8^-$ planes and 16 D8s.

String compactifications down to 8d

We obtain three possible $\mathcal{N}=1$ 8d theories by putting the previous $\mathcal{N}=1$ theories on an S^1. The resulting theories are neatly described in IIB language (on $T^2/(\mathcal{I}\Omega(-1)^{F_L})$):

- **Rank 4:** IIB with two $O7^-$ and two $O7^+$.
- **Rank 12:** IIB with three $O7^-$, one $O7^+$ and 8 D7s.
- **Rank 20:** IIB with four $O7^-$ and 16 D7s.

All these cases can also be described in F-theory, possibly with frozen singularities. (For a detailed discussion of the moduli spaces and dual pictures, see [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi '01] and [Taylor '11].)

Non-abelian enhancements

$\mathcal{N} = 1$ theories in 8d have a complex scalar in the vector multiplet. Giving a generic vev to these scalars costs no energy, and breaks the gauge algebra to $\mathfrak{u}(1)^{\mathrm{rk}}$. The set of all vacua accessed in this way is the *Coulomb branch*.

At certain points in the Coulomb branch there can be non-abelian enhancements. The enhancements in the known backgrounds are to $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{sp}(N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

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We would like to explain why the other algebras

$$\mathfrak{so}(2N+1) \quad ; \quad \mathfrak{f}_4 \quad \text{and} \quad \mathfrak{g}_2$$

do not appear.

Computing global anomalies in 8d

[I.G.-E., Hayashi, Ohmori, Tachikawa, Yonekura '17]

Ideally, we would

- Construct all the bordism groups $\Omega_9^{\mathrm{Spin}}(BG)$.
- Compute the $e^{2\pi i\,\eta}$ homomorphism.

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- Construct all the bordism groups $\Omega_9^{\text{Spin}}(BG)$.
- Compute the $e^{2\pi i\,\eta}$ homomorphism.

This is hard. We did the following instead:

- Choose $X_8 = S^4 \times \mathbb{R}^4$, and put an appropriate G bundle on S^4.
- See if the effective theory on $\mathbb{R}^4$ after reduction has a global (Witten) anomaly. $(\eta(A \times B) = \text{ind}(A) \cdot \eta(B))$

The resulting conditions are (in principle) weaker, but still illuminating!

Example: $\mathfrak{so}(2N+1)$ with $N > 2$

Up to discrete factors, any global form of this algebra has a subgroup

$$SU(2) \times SU(2)' \times SO(2N-3)\,. \tag{21}$$

We put the instanton on $SU(2)$, so the unbroken group in 4d is $H = SU(2)' \times SO(2N-3)$.

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The adjoint of $\mathfrak{so}(2N+1)$ decomposes as

$$\begin{gathered} \mathbf{2}_{SU(2)} \otimes \mathbf{2}_{SU(2)'} \otimes (\mathbf{2N-3})_{SO(2N-3)} \\ \oplus \mathrm{Adj}(SU(2) \times SU(2)' \times SO(2N-3)) \end{gathered} \tag{22}$$

The resulting representation in four dimensions of H is

$$r_H = \mathbf{2}_{SU(2)'} \otimes (\mathbf{2N-3})_{SO(2N-3)} + (H \text{ singlets}). \tag{23}$$

which manifestly has a global anomaly.

Summary of results

We find that 8d $\mathcal{N} = 1$ theories with algebra $\mathfrak{f}_4$ and $\mathfrak{so}(2N+1)$ for $N \geq 3$ do not exist quantum mechanically, due to an anomaly.

We find no anomaly for $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ and $\mathfrak{g}_2$.

We find no ordinary global anomaly for $\mathfrak{sp}(N)$ (associated to S^8), but there is an anomaly on $S^4 \times \mathbb{R}^4$.

- The $d = 8$ $\mathcal{N} = 1$ $\mathfrak{sp}(N)$ theories *are* inconsistent.
- But perhaps this inconsistency can be cured by coupling to a TQFT (the *topological Green-Schwarz mechanism*). We conjecture that this is what happens on the worldvolume of an $O7^+$.
- The needed TQFT is necessarily somewhat involved (it should involve K-theory instead of cohomology), and we have not been able to construct it.

Conclusions

The Dai-Freed viewpoint provides a mathematically precise formulation of what it means for a theory to be anomaly-free, and fits well with intuition from quantum gravity.

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I expect that there will be interesting applications to our understanding of the 8d/9d swampland, beyond the results in [I.G.-E., Hayashi, Ohmori, Tachikawa, Yonekura '17].

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I expect that there will be interesting applications to our understanding of the 8d/9d swampland, beyond the results in [I.G.-E., Hayashi, Ohmori, Tachikawa, Yonekura '17].

With Miguel, we have been looking instead to applications of these ideas to 4d theories, and more specifically to the Standard Model.

Supplementary material

A new anomaly for $\mathfrak{sp}(2N)$

Consider the decomposition $USp(2N) \supset USp(2) \times USp(2N-2)$. The adjoint decomposes as

$$\text{Adj} \to (\mathbf{2} \otimes (\mathbf{2N-2})) \oplus (\text{Adj} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \text{Adj})\,.$$

so the effective $USp(2N-2)$ theory in $\mathbb{R}^4$ has fermions in the representation

$$r_H = \square \oplus (\text{singlets})\,. \tag{24}$$

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So there is a Witten anomaly!