The Weak Gravity Conjecture in F-theory

- 1803.07998 w/ Seung-Joo Lee and Diego Regalado
- to appear w/ Seung-Joo Lee, Wolfgang Lerche, Diego Regalado

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Gravity and U(1)s

Quantum gravity effects deeply woven into fabric of string theory:
No arbitrary independent tuning of gravitational and gauge degrees of freedom possible

• Heterotic string: Gravity and gauge d.o.f. both in closed sector
• Open string/brane theories: Non-abelian gauge theory can be decoupled from gravitational bulk only when UV complete by itself - geometrically by shrinking brane cycles of non-abelian branes

$U(1)$ gauge theories special

• Field theory: by themselves not UV complete
• Geometry: Non-shrinkability of height pairing of rational sections in F-theory [Lee, Regalado, TW’18] see talk by S.-J. Lee

Deep interrelations between quantum gravity and string geometry
Gravity and U(1)s

**Lesson:**
For $U(1)$ gauge theory in F-theory, $M_{Pl} \rightarrow \infty$ only possible for $g_{YM} \rightarrow 0$

This talk considers the converse question:
What happens if take $g_{YM} \rightarrow 0$ at $M_{Pl}$ finite?

Field theory intuition:

1. **Weak Gravity Conjecture** [Arkani-Hamed,Motl,Nicolis,Vafa’06], …
   - Gravity is weakest force.

2. In presence of gravity, **no global symmetries**. [Banks,Dixon’88], …
   - All symmetries must be gauged in the UV.

General expectation: [Ooguri,Vafa’06], …

✓ Offensive limit should be at infinite distance (beyond reach)
✓ Effective theory must break down (**quantum gravity censorship**):
   - As $g_{YM} \rightarrow 0$ expect **infinitely many charged massless states**.
Gravity and U(1)s

In particular: [Grimm,Palti,Valenzuela’18] analyses this (see talk by I. Valenzuela)

- in context of 4d $N = 2$ Type IIB compactifications
- for $U(1)$ in question from Type IIB Ramond Ramond sector

[Palti’17] [Kl"awer,Palti’16] [Heidenreich,Reece,Rudelius’16/’18] [Andriolo,Junghans,Noumi,Shiu’18]
[Blumenhagen,Kl"awer,Schlechter,Wolf’18], . . . cf. talks by Corvilain, Kl"awer, Wolf, . . .

Hardly studied quantitatively so far:
Quantum Gravity Conjectures and $U(1)$/gauge symmetries in ’open string sector’

This talk:

- Quantitative study of various QG conjectures for ’open string’ $U(1)$s in F-theory
- For better control of quantum corrections to geometry: F-theory in 6d
- Results apply to abelian or non-abelian gauge groups alike
Summary of results

Main results: [Lee, Lerche, Regalado, TW to appear]

1. For fixed $M_{\text{Pl}}$, limit $g_{\text{YM}} \to 0$ lies at infinite distance in Kähler moduli space of base $B_2$.
2. As $g_{\text{YM}} \to 0$, necessarily tensionless strings arise in the 6d compactification.
   Math: Mori’s cone theorem
3. These give rise to infinitely many massless BPS particles which are charged under the gauge group.
   Math: Theory of weak Jacobi forms and elliptic genera
4. The Sublattice Weak Gravity Conjecture is satisfied (at least asymptotically) if one takes into account the (massive) states from strings wrapped on a D3-brane.

Formal new results:
Computation of elliptic genus for general Mordell-Weil flavour group
**F-theory in 6d**

**Context:** F-theory compactified on elliptic CY $3$ $Y_3 \to B_2$

Effective theory in $\mathbb{R}^{1,5}$:

$N = (1, 0)$ supergravity (8 SUSYs)

base $B_2$: complex surface

7-branes on complex curve $C \subset B_2$

**Couplings:**

$$M^4_{Pl} = 4\pi \text{vol}_J(B_2)$$

$$\frac{1}{g_{YM}^2} = 4\pi \text{vol}_J(C)$$

- **non-abelian gauge algebra** $\mathfrak{g}$:
  
  $C$ contained in discriminant of fibration (wrapped by brane stack)

- **abelian gauge algebra** $\mathfrak{g} = u(1)_A$: [cf talks by Cvetič, Mayorga, Dierigl]

  $C = -\pi_*(\sigma(S_A) \cdot \sigma(S_A))$ (height pairing of rational section $S_A$)
Global limit in Kähler geometry

[Aim:]

\[
\frac{1}{g_{YM}^2} \sim \text{vol}_J(C) \to \infty \quad \text{while} \quad M_{Pl} \sim \text{vol}_J(B_2) \equiv 1 \quad (*)
\]

[Result:]

There must exist another curve \( C_0 \) with

\[
C_0 \cdot C \neq 0 \quad \text{and} \quad \text{vol}_J(C_0) \to 0
\]

[General intuition]

"On finite volume surface, if one direction gets big, normal direction must get very small".

[Proof:]

**Step 1)** Limit \((*)\) requires asymptotically - (in simplest case!)

\[
J = a(t) \left( t J_0 + \frac{1}{2t} N \right) \quad \text{with} \quad t \to \infty, \quad a(t) = \left(1 + \frac{\gamma}{4t^2}\right)^{-1/2}
\]

- \( \int_{B_2} J_0 \cdot J_0 = 0, \quad \int_{B_2} J_0 \cdot N = 1, \quad \gamma = \int_{B_2} N \wedge N \)
- \( \int_{C} J_0 = 1 \)
Global limit in Kähler geometry

\[
\frac{1}{g_{\text{YM}}^2} \sim \text{vol}_J(C) \to \infty \quad \text{while} \quad M_{\text{Pl}} \sim \text{vol}_J(B^2) \equiv 1
\]

\[J \sim \left( tJ_0 + \frac{1}{2t}N \right) \quad \text{with} \quad \int_{B^2} J_0 \cdot J_0 = 0, \quad \int_{B^2} J_0 \cdot N = 1\]

1. W.l.o.g. write \( J = A_i J_i + \tilde{A}_i \tilde{J}_i + a_k' J'_k \)
   - \( J_i, \tilde{J}_i, J'_k \) Kähler cone generators
   - \( A_i, \tilde{A}_i \to \infty, a_k' \) finite \( \geq 0 \)
   - \( \int_C J_i \neq 0, \int_C \tilde{J}_i = 0 \)

2. Simplest case: Fastest rate of divergence for some \( A_i \) else - see paper!
   \[ \Rightarrow J = t(a_i J_i + \tilde{a}_i \tilde{J}_i) + a_k' J'_k =: tJ_0 + J_2 \quad \int_C J_0 \neq 0 \]

3. \( \text{vol}_J(B^2) = \int_{B^2} J^2 = \int_{B^2} \left(t^2 J_0^2 + J_2^2 + 2tJ_0J_2\right) \) finite as \( t \to \infty \)
   \[ \Rightarrow J_0^2 = 0 \text{ and } J_0 \cdot J_2 \sim \frac{1}{2t} \]
Global limit in Kähler geometry

Step 1) \( J \sim tJ_0 + \frac{1}{2t} N \) with \( \int_{B_2} J_0^2 = 0, \int_{B_2} J_0 \cdot N = 1 \)

Step 2)

Idea: If \( J_0 \) is the class of a holomorphic curve \( C_0 \), i.e. if

\[ J_0 \sim [C_0] \]

✓ \( \text{vol}_J(C_0) \sim \frac{1}{2t} \int_{C_0} N \to 0 \) as \( t \to \infty \)

✓ \( C_0 \cdot C \neq 0 \) because \( \int_C J_0 \neq 0 \)

Indeed this is the case, and in addition \( C_0 \) is a \( \mathbb{P}^1 \).

• \( J_0 \in \overline{K} \) (Kähler cone closure) \( \subset \overline{NE(B_2)} \) (Mori cone closure)

• Mori’s cone theorem: \( \overline{NE(B_2)} = \overline{NE(B_2)}_{\overline{K} \leq 0} + \sum_i \mathbb{R}_{\geq 0} [C_i] \) \( C_i \) are \( \mathbb{P}^1 \)s

• We can rule out \( J_0 \cdot \overline{K} \leq 0 \) for F-theory base:
  \( \overline{K} \geq 0 \) and exists \( n > 0 \): \( n\overline{K} \geq C \) with \( J_0 \cdot C > 0 \) for \( C \) a discriminant component or height pairing

\( \Rightarrow J_0 \) is effective curve class \( C_0 \). \( \text{Uses that } B_2 \text{ is F-theory base!} \)
Example

$B_2 = dP_1$: $\mathbb{P}^2$ (class: $H$) with one blowup curve $E$

$$H^2 = 1, \quad E^2 = -1, \quad H \cdot E = 0.$$ 

- Kähler cone generators: $H - E$, $H$
  \[ \Rightarrow J = a(H - E) + bH \text{ for } a \geq 0, \ b \geq 0 \]

- Mori cone generators: $H - E$, $E$

- $\text{vol}(H) = a + b$, $\text{vol}(E) = a \Rightarrow H$ large and $E$ large for $a$ large

- $\text{vol}(B_2) = J^2 = b^2 + 2ab$ finite as $a \sim t \to \infty$ only for $b \sim \frac{1}{t}$

  $\Rightarrow C_0 = H - E$ with $\text{vol}(C_0) \sim 1/t$
Tensionless Strings

- **7-brane on** \( C \)
  \[
  \text{vol}_J(C) \sim t \sim \frac{1}{g_{YM}^2} \to \infty
  \]

- **curve** \( C_0 \) **with** \( C_0 \cdot C \neq 0 \)
  \[
  \text{vol}_J(C_0) \sim \frac{1}{2t} \to 0
  \]

D3-brane wrapped on \( C_0 \) gives rise to string in \( \mathbb{R}^{1,5} \)

\( = \) non-critical string theory in 6d with tension \( T = (2\pi)\text{vol}_J(C_0) \)

**Tower of BPS excitations:** [Witten’97]

\[
M_n^2 \sim nT \quad n = 1, 2, 3, \ldots \quad \text{excitation level}
\]

**Maximal charge** \( q_n \) **per excitation level** \( n \): - derived later

\[
q_n^2 = \beta n \quad \beta = \mathcal{O}(1)
\]

As \( g_{YM} \to 0 \): the string becomes **tensionless** and

this **tower of charged states becomes massless**.
Quantum Gravity Conjectures

1) Since global symmetries are not possible, the limit must be at infinite distance in moduli space, i.e. it cannot be reached.

Indeed this is the case here.

Proof: For general surface $B_2$

- Basis $\omega_\alpha \alpha = 1, \ldots, h^{1,1}(B_2) = d$, $\omega_\alpha \cdot \omega_\beta = \text{diag}(+1, -1, \ldots, -1)$
- Kähler form $J = j_\alpha \omega_\alpha$ $J^2 = 1$

  $\Rightarrow j^0 = \cosh x$, $j^i = (\sinh x) u^i(\phi_1, \ldots, \phi_{d-2})$ with $\sum_i (u^i)^2 = 1$

- Metric on Kähler moduli space:

  $$ds^2 = dx^2 + (\sinh x)^2 d\Omega_{S^{d-2}}$$

- Distance between points $P$ and $Q$ in moduli space:

  $$d(P, Q) = \int \sqrt{1 + (\sinh x)^2 h_{AB}\phi'^A\phi'^B} dx \geq x_P - x_Q$$

Present case: $t \sim \frac{1}{2}e^x$ \Rightarrow Limit $t \rightarrow \infty$ at distance $\Delta \sim \log(t)$
Quantum Gravity Conjectures

1) No global symmetries.
   ⇒ The limit must be at infinite distance in moduli space.
   Indeed this is the case here.

   Conclusion: Limit $t \to \infty$ at distance $\Delta \sim \log(t) \to \infty$

2) Swampland Distance Conjecture: [Ooguri,Vafa] [Klaewer,Palti’16],
   [Grimm,Palti,Valenzuela’18] [Heidenreich,Reece,Rudelius’17,’18]
   [Andriolo,Junghans,Noumi,Shiu’18]
   Infinitely many - charged! - states become massless at exponential rate.
   Present case:

   $\Delta \sim \log(t), \quad m_n^2 \sim n T \sim n \text{vol}_J(C_0) \sim \frac{n}{2t} \sim e^{-\Delta}$
Quantum Gravity Conjectures

3) Sublattice Weak Gravity Conjecture [Heidenreich, Reece, Rudelius’16-’18]

\[ Q^2 g_{YM}^2 \geq kM^2G_N = \frac{k}{8\pi} \frac{M^2}{M_{Pl}^4}, \]

Present case: [Lee, Lerche, Regalado, TW to appear]

- Maximal charge per excitation level: \( q_n^2 = \beta n \)
- \( m_n^2 = \alpha n T = 2\pi \alpha n \text{vol}(C_0) = 2\pi \alpha n a(t) \frac{1}{2t} \)
- \( \frac{1}{g_{YM}^2} = 4\pi \text{vol}C = 4\pi a(t)t \)

\[ q_n^2 g_{YM}^2 = \frac{\beta}{\pi \alpha} A(t) \frac{m_n^2}{M_{Pl}^4} \quad A(t) = 1 + \frac{\gamma}{4t^2} \rightarrow 1 \]

Note: All charges are present and satisfy the quantum completeness conjecture of [Polchinski’03].
Non-critical strings

• 7-brane on $C$

• D3-brane on curve $C_0$ with
  $C_0 \cdot C \neq 0$
  $\Rightarrow$ non-critical string on curve
  $C_0$ with $C_0^2 = 0$, $C_0 \cdot \bar{K} = 2$

Twisted reduction of $N = 4$ SYM with varying gauge coupling along $C_0$

[Martucci’14][Haghighat,Murthy,Vafa,Vandoren’15][Lawrie,Schafer-Nameki,TW’16] talk by Mayer

• 2d $N = (0, 4)$ effective theory describes worldsheet theory

• At intersection $C_0 \cap C$:
  isolated 3-7 string modes \textbf{charged} under 7-brane gauge group
  $\Rightarrow$ gauge group on $C$ becomes global/flavour symmetry group of
  non-critical string
  $\Rightarrow$ excitations of string will be charged under 7-brane group
The elliptic genus

A certain index of resulting BPS states is counted by elliptic genus:
Consider string on a torus $T^2$ with periodic (R) boundary conditions:

$$Z_{C^\beta}(\tau, \lambda_s, z_a) = \text{Tr}_R(-1)^F q^H L q^H R u^{2J} \prod_a (y_a)^J a$$

$q = e^{2\pi i \tau}$: $\tau$ complex structure of $T^2$

$u = e^{2\pi i \lambda_s}$: fugacity w.r.t. $SU(2) \supset SO(4)_{T}$ (6d spin)

$y_a = e^{2\pi i z_a}$: fugacity w.r.t. flavour symmetry Cartan $U(1)_a$

$Z_{C^\beta}$ is a Weyl invariant Jacobi form of weight $w = 0$

fugacity index $m_u$ and $m_ya$ determined by the geometry

$$\varphi_{w,m} \left( \frac{a\tau + b}{c\tau + d}, \frac{\zeta}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi i \frac{m u c}{c\tau + d} \frac{\zeta, \zeta}{2}} \varphi_{w,m}(\tau, \zeta)$$

$$\varphi_{w,m}(\tau, \zeta + \lambda \tau + \mu) = e^{-2\pi i m \left( \frac{\zeta, \zeta}{2} \right) \tau + 2 \frac{\lambda, \zeta}{2}} \varphi_{w,m}(\tau, \zeta)$$
The elliptic genus

\[ Z_{C_\beta}(\tau, \lambda_s, z_a) = \text{Tr}_R(-1)^F q^{H_L} q^{H_R} u^{2J} \prod_a (y_a)^J_a \]

\[ = \sum c(n, r_u, r_a) q^n u^{r_u} \prod_a y_a^{r_a} \]

\[ \Rightarrow c(n, r_u, r_a) \] index of BPS states at excitation level \( n \) and \( U(1)_a \) flavour charge \( r_a \)

\( Z_{C_\beta} \) studied intensively for shrinkable curves \( C_\beta \cdot C_\beta = -1, -2, \ldots, -12 \)

For extensive 6d literature - see review [Heckman,Rudelius’18]

[Klemm,Mayr,Vafa’96] [Klemm,Manschot,Wotschke’12] [Alim,Scheidegger’12]
[Benini,Eager,Hori,Tachikawa’13] [Haghighat,Lockhart,Vafa’14] [Cai,Huang,Sun’14]
[Haghighat,Klemm,Lockhart,Vafa’14] [Huang,Katz,Klemm’15] [Haghighat,Murthy,Vafa,Vandroen’15]
[Kim,Kim,Lee’15] [DelZotto,Lockhart’16/’18] [Gu,Huang,Kashani-Poor,Klemm’17]
[Kim,Lee,Park’18],...
The elliptic genus

Novelty: $C_0 \cdot C_0 = 0$ and with general $U(1)_A$ flavour symmetries

- Ansatz from analysis of pole structure: [Klemm et al. -’12-'18], [Vafa et al.]

$$Z_{C_0}(\tau, \lambda_s, z) = \left( \frac{1}{\eta^2(\tau)} \right)^{6C_0 \cdot \tilde{K}} \frac{\Phi_{W,L,m}(\tau, \lambda_s, z)}{\varphi_{-2,1}(\tau, \lambda_s)}.$$  

$W = 6C_0 \cdot \tilde{K} - 2 = 10$,  
$L = \frac{1}{2} C_0 \cdot (C_0 + K) + 1 = g(C_0) = 0$

- Fix fugacity index w.r.t. $U(1)_A$ [Lee,Lerche,Regalado,TW to appear]

$$m_A = \frac{1}{2} C_0 \cdot h_A \quad h_A = -\pi_*(\sigma(S_A) \cdot \sigma(S_A)) \quad \text{(height pairing)}$$

- General ansatz for $\Phi_{W,L,m}(\tau, \lambda_s, z)$ in terms of products of $E_4(\tau), E_6(\tau), \varphi_{-2,1}(\tau, z), \varphi_{0,1}(\tau, z)$

- General properties of Eisenstein-Jacobi forms imply for maximal $U(1)_A$ charge per excitation level:

$$(q_A)_{n}^{2} = \beta n \quad \text{for} \quad \beta \geq m_A$$
The elliptic genus

Compare with known BPS numbers via mirror symmetry computation of BPS invariants using duality with M-theory to fix ansatz completely

\[ \begin{align*}
6d \text{ F-theory on } S^1 & \quad 5d \text{ M-theory} \\
\text{string wrapped on } S^1 & \quad \text{BPS particle in 5d} \\
\text{wrapping number } m & \quad \text{M2 brane on} \\
\text{KK momentum } k & \quad mC_\beta + kE_\tau
\end{align*} \]

Relation: \( mk = n \) \[\text{[Klemm, Mayr, Vafa’96]}\]

⇒ Counting of excitation level \( n \) in 6d equivalent to counting M2-brane states in 5d via Gopakumar-Vafa BPS invariants

Encoded in free energy of topological string on same CY \( Y_3 \):

\[ \mathcal{F}(\lambda_s, \tau, t, z) = \sum_{g \geq 0} \mathcal{F}(g)(\tau, t, z) \lambda_s^{2g-2} \]

\[ Z_{\text{top}} = \exp(\mathcal{F}(\lambda_s, \tau, t, z)) = Z_0(\tau, \lambda_s) \left( 1 + \sum_{C_\beta} Z_{C_\beta}(\tau, \lambda_s, z) e^{2\pi i t_\beta + \frac{1}{2}(C_\beta \cdot \bar{K}) \tau} \right) \]

[Haghighat et al.’13] [Haghighat, Klemm, Lockhart, Vafa’14],...
Non-critical strings

Explicit computation of elliptic genus and hence BPS spectrum for general $U(1)$ models of Morrison-Park type on Hirzebruch and dP$_2$ surfaces

Example: $B_2 = \mathbb{F}_{a=1}$

topological types of MP model specified by integer
$0 \leq x \leq 4$, $0 \leq y \leq 4 + 2a = 6$

$$(x, y) = (4, 6) : U(1)_A \text{ fugacity } m_A = 2$$

$$Z_{C_0} = \frac{\mathcal{F}_{C_0}^{(0)}(\tau, \zeta)}{\varphi_{-2,1}(\tau, \lambda_s)}$$

with

$$\mathcal{F}_{C_0}^{(0)}(\tau, \zeta) = \frac{q}{\eta^{24}} \left( -\frac{1}{72} E_4^2 E_6 \varphi_{-2,1}^2 + \frac{1}{54} E_4^3 \varphi_{-2,1} \varphi_{0,1} + \frac{1}{108} E_6^2 \varphi_{-2,1} \varphi_{0,1} - \frac{1}{72} E_4 E_6 \varphi_{0,1}^2 \right)$$

$$= -2 + \left( \frac{96}{\zeta} + 288 + 96\zeta \right) q$$

$$+ \left( -\frac{2}{\zeta^4} + \frac{96}{\zeta^3} + \frac{10192}{\zeta^2} + \frac{69280}{\zeta} + 123756 + 69280\zeta + 10192\zeta^2 + 96\zeta^3 - 2\zeta^4 \right) + \mathcal{O}(q^3).$$
Mass-Charge relation

$r_n$: maximal charge per excitation level: $r_n^2 = \beta n$

Proven lower bound: $\beta \geq m_A = 2$ (fugacity index)

In fact: closer to $\beta \sim 4m_A$ in all studied examples
Summary

Systematic study of Quantum Gravity constraints for open string U(1)s

1. Gauge group become global at infinite distance.

2. $g_{YM} \rightarrow 0$ implies tensionless non-critical strings.

3. Infinite tower of charged BPS states becomes massless, in agreement with
   - Swampland Distance Conjecture
   - Sublattice Weak Gravity Conjecture
   - Completeness Hypothesis

4. Explicit and quantitative analysis of elliptic genus in presence of general U(1)s

Weak Jacobi forms imply the weak gravity conjecture.