Non-Perturbative Superpotentials and Associatives

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String Phenomenology 2018, University of Warsaw

July 6, 2018

Based on work in collaboration with B. Acharya, A. Braun and R. Valandro

Introduction

Motivation and Overview

String model building often leaves a number of moduli to be "stabilised".

Perturbative effects and fluxes often fall short $\Rightarrow$ Need *non-perturbative effects*, cf. talk by Ovrut and KKLT.

By wrapping branes on calibrated cycles one can produce such effects. F-theory: $M5$-instantons wrapping effective divisors in a Calabi-Yau four-fold [Donagi-Grassi-Witten '96] (DGW).

String dualities $\Rightarrow$ Should be effects dual to DGW in the other string/M-theories. Heterotic: World-sheet instantons [Curio-Lüst '97, Anderson etal '15]. M-theory: Euclidean $M2$-branes [Braun etal '18]. Can lead to new *interesting mathematical conjectures*.

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Overview:

- Review of non-perturbative superpotential in F-theory as described by DGW.
- The weak coupling Sen limit and Type IIB.
- Mirror symmetry and special Lagrangians of the type IIA Calabi-Yau.
- Conclusions and Outlook: The M-theory lift to and associative sub-manifolds of G_2 geometry.

Some Calibrated Geometry

Given (M, Φ), where Φ is some p-form denoting some extra structure associated to M (e.g. CY, G_2, etc where we also insist that $\mathrm{d}\Phi = 0$).

Φ is a calibration if for $x \in M$ we have $\Phi_x = \lambda \mathrm{vol}_\xi$ where $\lambda \leq 1$ $\forall$ oriented $\xi \subseteq T_x M$.

A p-dimensional sub-manifold $N \subseteq M$ is calibrated w.r.t. Φ if $\Phi|_N = \mathrm{vol}_N$. We then have

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N = \int_N \Phi = \int_{\tilde{N}} \Phi \leq \int_{\tilde{N}} \mathrm{vol}_{\tilde{N}} = \mathrm{Vol}(\tilde{N}) \,,$$

where N and $\tilde{N}$ are in the same homology class.

In this sense, calibrated sub-manifolds are *minimal surfaces*. BPS conditions $\Rightarrow$ Can wrap branes on minimal surfaces.

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Examples:

- Kähler manifold: Normalised powers of the Kähler form. Calibrated sub-manifolds are complex submanifolds, e.g. *effective divisors*.
- Calabi-Yau: The real part of a holomorphic volume form. Calibrated submanifolds are *special Lagrangian*.
- G_2: The associative/co-associative three- and four-form. Calibrated submanifolds are *associative* and *co-associative*.

Review of Donagi-Grassi-Witten

The argument of DGW relies on counting sections of an elliptic fibration of a rational elliptic surface (a dP_9). DP_9's second cohomology forms the lattice

$$H^2(dP_9, \mathbb{Z}) = H^{(1,1)}(dP_9, \mathbb{Z}) = -E_8 \oplus U = -E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} .$$

Fiber class: $F \in U$ s.t. $F^2 = 0$. *Zero section:* $\sigma_0 \in U$ s.t. $\sigma_0^2 = -1$.

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A section σ of the elliptic fibration corresponding to an irreducible P_1 satisfy

$$F \cdot \sigma = 1$$
$$\sigma^2 = -1 \, .$$

These span a nine-dimensional subspace of $H_2(dP_9,\mathbb{Z}) \cong H^{(1,1)}(dP_9,\mathbb{Z})$. Indeed, given $\gamma \in -E_8$ s.t. $\gamma^2 = -2n$, we can take

$$\sigma_\gamma = \gamma + \sigma_0 + nF \, .$$

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DGW: 1-1 correspondence between σ_γ's and E_8 lattice points. Give rise to effective divisors in CY_4. Wrap 5-branes $\Rightarrow$ non-perturbative superpotential parametrised by an E_8 theta-function.

Sen Limit and Type IIB

The Type IIB Calabi-Yau

F-theory fourfold is given by an elliptic fibration over $B = dP_9 \times P_1$. The effective divisors of DGW are elliptic fibrations over $P_1 \times P_1$, where the first P_1 are given by the sections above.

Sen limit $\Rightarrow$ type IIB Calabi-Yau X as a double cover of B, *the split bicubic* (Schoen). Type IIB superpotential is given by Euclidean $D3$-branes wrapping a double cover of $P_1 \times P_1$, a dP_9. These lift to the $M5$-instantons of DGW.

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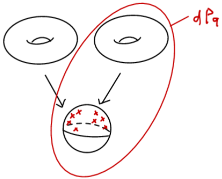
One can view the split bicubic as a double elliptic fibration over P_1:



Choosing a section of one elliptic fibration $\Rightarrow$ type IIB dP_9 divisor. There is another set of divisors corr. to the other fibration, but *not invariant* under the orientifold involution.

The Orbifold Limit

The bicubic has a Voisin-Borcea orbifold representation [Voisin '92, Borcea '92, '97]:

$$X = K3 \times T^2/\mathbb{Z}_2 \, ,$$

$\mathbb{Z}_2$ acts as minus the identity on T^2 and as the Nikulin involution $(10, 8, 0)$ on $K3$. Note that $dP_9 \cong K3/\mathbb{Z}_2$. In this limit the divisors have the form $P_1 \times T^2/\mathbb{Z}_2$, where P_1 is a section of the $K3$ elliptic fibration.

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The Nikulin involution acts on $H^2(K3, \mathbb{Z}) = -E_8^+ \oplus -E_8^- \oplus U_1 \oplus U_2 \oplus U_3$ as

$$\mathbb{Z}_2 \circlearrowleft \frac{\begin{array}{ccccc|c} E_8^+ & E_8^- & U_1 & U_2 & U_3 & z \end{array}}{\begin{array}{ccccc|c} E_8^- & E_8^+ & U_1 & -U_2 & -U_3 & -z \end{array}} \,, \quad U_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

where z is the coordinate on T^2 and the U_i's are hyperbolic lattices. The Calabi-Yau structure forms are given by

$$\omega_X = \omega_{K3} + \tfrac{i}{2} dz \wedge d\bar{z} \,, \quad \omega_{K3} \in \Lambda^+ \otimes \mathbb{R} \,, \quad \Lambda^+ = \langle e_1, e^1, \alpha_i^+ + \alpha_i^- \rangle$$
$$\Omega_X^{3,0} = \Omega_{K3}^{2,0} \wedge dz \,, \quad \Omega_{K3}^{2,0} \in \Lambda^- \otimes \mathbb{C} \,, \quad \Lambda^- = \langle e_2, e^2, e_3, e^3, \alpha_i^+ - \alpha_i^- \rangle$$

Mirror Symmetry and Type IIA

The Type IIA Mirror

The bicubic is self-mirror, and as a Voisin-Borcea orbifold takes again the form

$$X^\vee = K3^\vee \times T^2/\mathbb{Z}_2^\vee ,$$

where $K3^\vee$ is the mirror of $K3$ and $\mathbb{Z}_2^\vee$ is the dual orbifold. Mirror symmetry on $K3$ acts as a hyper-Kähler rotation [Aspinwall '96, Gross '98]

$$\begin{aligned} \omega_{K3} &\to \mathrm{Im}(\Omega^{2,0}_{S^\vee}) \\ \mathrm{Im}(\Omega^{2,0}_{K3}) &\to -\omega_{K3^\vee} \\ \mathrm{Re}(\Omega^{2,0}_{K3}) &\to \mathrm{Re}(\Omega^{2,0}_{K3^\vee}) \,. \end{aligned}$$

To get a Calabi-Yau three-fold, we find that the dual orbifold must act as

$$\mathbb{Z}_2^\vee \circlearrowleft \begin{array}{ccccc|c} E_8^+ & E_8^- & U_1 & U_2 & U_3 & z \\ \hline -E_8^- & -E_8^+ & -U_1 & -U_2 & U_3 & -z \end{array} .$$

Special Lagrangians and Type IIA Superpotential

Under mirror symmetry, the $D3$ branes wrapping divisors become Euclidean $D2$ branes warpping special Lagrangian (sLag) three-cycles. This can be understood by following three T-dualities of the SYZ fibration of X.

For a special Lagrangian C, we require

$$C \cdot \mathrm{Im}(\Omega_X^{3,0}) = \int_C \left(\mathrm{Im}(\Omega_X^{3,0})\right) = \int_C \left(\mathrm{Re}(\Omega_{S^\vee}^{2,0}) \wedge dy + \mathrm{Im}(\Omega_{S^\vee}^{2,0}) \wedge dx\right) = 0\,,$$

where $z = x + iy$.

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Under the type IIA orientifold, the holomorphic top-form is required to transform as

$$\Omega_{X^\vee}^{3,0}=\Omega_{K3^\vee}^{2,0}\wedge dz\to\overline{\Omega_{K3^\vee}^{2,0}}\wedge d\bar z=\overline{\Omega_X^{3,0}}\,.$$

This forces us to take

$$\begin{aligned}\mathrm{Re}(\Omega_{K3^\vee}^{2,0})&\in\langle e_2,e^2\rangle\otimes\mathbb{R}\\ \mathrm{Im}(\Omega_{K3^\vee}^{2,0})&\in\langle e_1,e^1,\alpha_i^++\alpha_i^-\rangle\otimes\mathbb{R}\,.\end{aligned}$$

Special Lagrangians and Type IIA Superpotential

The sLag's dual to type IIB divisors are given by an odd rational curve σ_γ on $K3^\vee$ times an odd one-cycle of T^2 modulo the orbifolding

$$C_\gamma = \sigma_\gamma \times \mathbb{S}_y / \mathbb{Z}_2^{\beta\vee} \ .$$

C_γ have topology of three-sphere.

$$\sigma_\gamma = \sigma_0 + 2nF + \gamma^+ + \gamma^- \ ,$$

where $\sigma_0 = e_1 - e^1 \ , \quad F = e^1$, and $\gamma^\pm$ are two identical copies in $E_8^\pm$ of the same element γ in E_8 such that $\gamma^2 = -2n$.

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where $\sigma_0 = e_1 - e^1$, $F = e^1$, and $\gamma^\pm$ are two identical copies in $E_8^\pm$ of the same element γ in E_8 such that $\gamma^2 = -2n$.

- $\sigma_\gamma^2 = -2$, necessary to represent a rational curve. Indeed, they correspond to sections of the elliptic fibration of the type IIB $K3$.
- Can check that $C_\gamma \cdot \mathrm{Im}(\Omega_X^{3,0}) = 0 \Rightarrow C_\gamma$'s are indeed special Lagrangian.
- Resolving orbifold singularities does not alter the complex structure $\Rightarrow$ expect C_γ's to remain sLag after resolution. Also expect resolution to introduce an additional E_8 lattice worth of sLag's, cf. [Curio-Lüst '97].
- Get an infinity of sLag's in bicubic parametrised by E_8 lattice.

Outlook

Conclusions and Outlook

Conclusions and Outlook: Lift to M-theory

Conclusions:

- We have followed the F-theory DGW superpotential to the weak coupling type IIB limit, and to type IIA through mirror symmetry.
- We have found the divisors wrapped by $D3$ branes of the type IIB Calabi-Yau. These are parametrised by an E_8 lattice, and are mapped to sLag's under mirror symmetry.
- We conjecture that the bicubic contains infinitely many sLag's, parametrised by an E_8 lattice.

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Outlook and work in progress:

- The sLags will lift to *associative submanifolds* in an M-theory G_2 geometry.
- We want to compare and contrast these against the associative cycles found by [Braun etal '18].

Thank you for your attention!