Group Theory Aspects of Non-Supersymmetric Heterotic Partition Functions

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in collaboration with
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\textsuperscript{1} at this conference
SUSY breaking string models have received a lot of interest in the past decades.

In particular, there is the $\text{SO}(16) \times \text{SO}(16)$ heterotic string with $\mathcal{N} = 0$ in 10 dimensions.

Alternatively, one may break SUSY during compactification: e.g. Scherk–Schwarz, or particular orbifold geometries.
Introduction

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- In particular, there is the SO(16) × SO(16) heterotic string with \( \mathcal{N} = 0 \) in 10 dimensions
- Alternatively, one may break SUSY during compactification: e.g. Scherk–Schwarz, or particular orbifold geometries

In general, heterotic non-SUSY string vacua suffer from a bunch of problems, such as a too large cosmological constant and instabilities
Cosmological constant/dilaton tadpole

As one can see from general arguments, the non-SUSY heterotic partition function \( Z(\tau, \bar{\tau}) \) is nonzero

\[
\Lambda \sim \int \frac{d^2 \tau}{\tau_2^2} \, Z(\tau, \bar{\tau})
\]

With the most prominent contribution coming from \textit{off-shell tachyons}

Abel, Dienes, Mavroudi '15

Dienes '90
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There are various proposals for this to vanish

- (space-time) Supersymmetry
- (generalized) Atkin-Lehner symmetry
Heterotic Orbifolds

Orbifolds can be obtained in two subsequent steps

- define and mod out a lattice \( \{ e_\alpha \}_{\alpha=1,...,D} \) to obtain a torus
- mod out a discrete isomorphism of the lattice (point group \( P \))

For Abelian point groups

\[
\text{SO}(6) \supset D_6(g) = \begin{pmatrix} e^{2\pi i v_1 J_{12}} & & \\ & e^{2\pi i v_2 J_{34}} & \\ & & e^{2\pi i v_3 J_{56}} \end{pmatrix} \quad \rightarrow \quad v_g = (v_1, v_2, v_3)
\]

\( \mathcal{N} = 1 \) SUSY \( \leftrightarrow \sum v_i = 0 \mod 2 \)
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Heterotic Orbifolds arise by an embedding of the geometric rotations and translations into the \( E_8 \times E_8 \) d.o.f.
The Heterotic Partition Function

The partition function can be organized

\[ Z = \sum_{g,h} Z \left[ \begin{array}{c} g \\ h \end{array} \right] \quad \text{where } g, h \in \text{ space group} \]

where we have to provide that \( g \) and \( h \) commute
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\[ \mathcal{Z} = \sum_{g,h} \mathcal{Z}[gh] \quad \text{where } g, h \in \text{ space group} \]

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It is known (via generalizations of the Jacobi Abstruse Identity):

\[ \mathcal{Z}[gh] \text{ vanishes } \Leftrightarrow \quad g \text{ and } h \text{ have common Killing spinor} \]
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However, the unbroken SUSY transformations need not be the same for all $(g, h)$ pairs
Local “SUSY enhancement”: An $\mathcal{N} = 2$ Example

A $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with roto-translation

$$v_\theta = \left(0, \frac{1}{2}, -\frac{1}{2}\right), \quad v_\omega = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$$

with generators

$$g_\theta = \left(\theta \left| \frac{1}{2} e_2\right)\right), \quad g_\omega = (\omega \left| 0\right)$$
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Fischer, Ratz, Torrado, Vaudrevange '12

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with generators

$$g_\theta = \left(\theta \mid \frac{1}{2}e_2\right), \quad g_\omega = (\omega \mid 0)$$

Because of the roto-translation, $[g_\theta, g_\omega] \neq 0$, and hence

$$\mathbb{Z} \not\ni \mathbb{Z} \left[\begin{array}{c} g_\theta \\ g_\omega \end{array}\right]$$

(the same holds for all SL(2, $\mathbb{Z}$) images)

$\mathcal{N} \geq 2$ SUSY in each $(g, h)$-twisted sector,

but

$\mathcal{N} = 1$ in the intersection!

Q: Can one achieve the same for $\mathcal{N} = 0 \rightarrow \mathcal{N} \geq 1$?
Four dimensional spinor representation

On the space-time fermions, each $D_6(g) \in 6$ of $SO(6)$ corresponds to a $D_4(g) \in 4$

Because $Spin(6) \cong SU(4)$

4 needs to be in $SU(4)$, i.e. $\det D_4(g) = 1$ for all $g$

Condition 1
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Caveats

- Elements of the geometric space group have more than one embedding into spinor space $\rightarrow$ Witten twist
- Possible degeneracies lead to spinor embeddings that are non-isomorphic to the geometric point group

For our considerations: do not care too much about whether spinor embedding corresponds to geometric embedding (just look at all possible actions on spinor space)
Realizing $\mathcal{N} = 0$

Every element of the point group commutes with the identity, therefore the untwisted partition function is given by

$$Z_{\text{untw.}} = \sum_{g \in P} Z \begin{bmatrix} 1 \\ g \end{bmatrix}$$

If we insist on SUSY breaking, then the projector of the untwisted sector must have trace 0

$$\sum_{g \in P} \text{tr} \, D_4(g) = 0$$
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or in terms of the representation content

4 does not contain a trivial singlet of $P$

Condition 2
Interplay: Modular orbits vs. projectors

Let us for the moment forget about the space group and consider the point group only. Then, all constructing elements $g$ are just elements of a finite group, i.e. $\exists N$ s.t. $g^N = 1$

$$
\mathbb{Z} \begin{bmatrix} 1 \\ g \end{bmatrix} \xrightarrow{S} \mathbb{Z} \begin{bmatrix} g \\ 1 \end{bmatrix} \xrightarrow{T} \mathbb{Z} \begin{bmatrix} g \\ g \end{bmatrix} \xrightarrow{T^{N-1}} \mathbb{Z} \begin{bmatrix} g \\ 1 \end{bmatrix}
$$

There may exist sectors that are not connected to these orbits, e.g. in Abelian $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds.

We always assume these sectors can be removed.
Conditions for $\mathcal{N} \geq 1$ locally

We are effectively dealing with projectors of the type

$$\sum_{n=0}^{N-1} Z \begin{bmatrix} g \\ g^n \end{bmatrix} \xrightarrow{\text{in group theory language}} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{tr} \, D_4(g^n)$$
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In representation theory language, this amounts to requiring that for all $g$

4 branches to the trivial singlet of every $\mathbb{Z}_N^{(g)}$ subgroup

Condition 3
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Then our rationale becomes

$$Z \begin{bmatrix} 1 \\ g \end{bmatrix} \quad \text{and entire modular orbit vanishes} \iff g \text{ preserves at least one Killing spinor} \iff 4 \text{ branches to trivial singlet } 1_0 \text{ of } \mathbb{Z}_N^{(g)}$$
A group-theoretical conjecture

**Conjecture.** For a given discrete group $G$, there does not exist a four dimensional representation with the properties that

(i) it has determinant 1,

(ii) it does *not* contain a *trivial singlet of $G* ,

(iii) to every $\mathbb{Z}_N$ subgroup of $G$ it branches into at least one *trivial singlet of that $\mathbb{Z}_N$ subgroup.*
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We have tested this conjecture for all point groups relevant for toroidal orbifolds, and for $\mathcal{O}(100\,000)$ more discrete groups.

It is crucial to insist on $\det = 1$ and dimension 4!
A chance for (generalizations of) Atkin–Lehner symmetry?

**Idea:** if $\mathcal{Z}(\tau, \bar{\tau}) \neq 0$, it might still integrate to zero

- Guaranteed if e.g. $\mathcal{Z}(-\frac{1}{N\tau}, -\frac{1}{N\bar{\tau}}) = -\mathcal{Z}(\tau, \bar{\tau})$ known as Atkin–Lehner symmetry Moore ’87

- does not seem to work (at least not with four non-compact dimensions) TR Taylor ’87, Balog and Tuite ’87

**but:**

- We see that a large portion of the $\mathcal{Z} \begin{bmatrix} 1 \\ g \end{bmatrix}$ vanishes even if $N = 0$ globally

- Hence, not the entire partition function has to have a certain symmetry
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work in progress
Conclusions

- we have translated the vanishing condition for \((g, h)\)-twisted sectors to a condition for the branching of the 4 to \(\mathbb{Z}_N\) subgroups
- we have formulated (and tested) a general group-theoretic conjecture for a No-Go
- the partial results point towards alternative solutions to the cosmological constant problem in (heterotic) non-SUSY strings
Conclusions

- we have translated the vanishing condition for \((g, h)\)-twisted sectors to a condition for the branching of the \(4\) to \(\mathbb{Z}_N\) subgroups
- we have formulated (and tested) a general group-theoretic conjecture for a No-Go
- the partial results point towards alternative solutions to the cosmological constant problem in (heterotic) non-SUSY strings

Thank You!