Strings on Celestial Sphere

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St.St., T.R. Taylor:

Strings on Celestial Sphere

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+ work to appear
Recap: from study of symmetries of scattering amplitudes:
   deep connections between
gravity and gauge interactions
   e.g.: KLT, BCJ, EYM (double-copy-construction)
   
   (in momentum or twistor space)
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traditional momentum space description:

\[ p_k^\mu, \quad k = 1, \ldots, N \]

\[ p_k^2 = - m_k^2 \]

- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates
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- amplitudes specified by asymptotic wave functions,
  which transform simply under space-time translations
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  between momentum eigenstates

D=4 Minkowski probably not the right space
to see all symmetries
of scattering amplitudes
Lorentz group in $\mathbb{R}^{1,D+1}$ is identical to Euklidian D-dimensional conformal group $SO(1,D+1)$

Scattering amplitudes in $\mathbb{R}^{1,D+1}$
interpretation
as Euklidian D-dimensional conformal correlators
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$D=2$: celestial sphere
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D=2: celestial sphere

Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections?
N particles on celestial sphere

\[ p_k \longrightarrow (E_k, z_k, \overline{z}_k) \]

represent points \( z_k \) on \( CS^2 \)

with:

\[ z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3} \quad , \quad E_k = p_k^0 \quad , \quad (\vec{p}_k)^2 = (p_k^0)^2 \]
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\[ p_k^\mu = E_k \left( 1, \frac{z_k + \bar{z}_k}{1 + |z_k|^2}, \frac{-i(z_k - \bar{z}_k)}{1 + |z_k|^2}, \frac{1 - |z_k|^2}{1 + |z_k|^2} \right) \]

\[ \omega_k = \omega_k q_k^\mu \]

\[ \omega_k = \frac{2 E_k}{(1 + |z_k|^2)} \]
N particles on celestial sphere

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\[ p_k^\mu = E_k \left( 1, \frac{z_k + \bar{z}_k}{1 + |z_k|^2}, \frac{-i(z_k - \bar{z}_k)}{1 + |z_k|^2}, \frac{1 - |z_k|^2}{1 + |z_k|^2} \right) \]

\[ := \omega_k \ q_k^\mu \]

\[ \omega_k = \frac{2 \ E_k}{(1 + |z_k|^2)} \]

Lorentz symmetry:

\[ z \rightarrow \frac{az + b}{cz + d} \]

global conformal symmetry on \( CS^2 \)
Amplitudes = conformal correlators of primary fields on $CS^2$

$$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3}$$

$$D = 4$$

$$D = 2$$

$$\sim \frac{g}{|z_1 - z_2|^{h_1 + h_2 - h_3} |z_2 - z_3|^{h_2 + h_3 - h_1} |z_1 - z_3|^{h_1 + h_3 - h_2}}$$
Amplitudes = conformal correlators of primary fields on $CS^2$

$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3}$

$D = 4$ space-time QFT correlators
$D = 2$ Euklidian CFT correlators

$D = 2$ CFT correlators involve conformal wave packets
Mellin transformation

\[ \tilde{\phi}(\Delta) = \int_0^\infty d\omega \, \omega^{\Delta-1} \phi(\omega) \]

In the massless case, with or without spin, the transition from momentum space to conformal primary wavefunctions with \( \Delta_j \) is implemented by Mellin transform
In practice: in momentum basis: plane waves with momentum $p$

conformal basis: conformal primary wave functions $\Delta$

Mellin transformation

\[
\tilde{\phi}(\Delta) = \int_0^\infty d\omega \ \omega^{\Delta-1} \ \phi(\omega)
\]

In the massless case, with or without spin, the transition from momentum space to conformal primary wavefunctions with $\Delta_j$ is implemented by Mellin transform

\[
\mathcal{A}(\{p_i, \xi_j\}) = i(2\pi)^4 \ \delta^{(4)} \left( p_1 + p_2 - \sum_{k=3}^{N} p_k \right) \mathcal{M}(\{p_i, \xi_j\})
\]

Mellin transform, with: $\Delta_j = 1 + i\lambda_j$

\[
\tilde{\mathcal{A}}(\lambda_n)(z_n, \bar{z}_n) = \left( \prod_{n=1}^{N} \int_0^\infty \omega_n^{i\lambda_n} \ d\omega_n \right) \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k) \times \mathcal{M}(\omega_n, z_n, \bar{z}_n)
\]
Three-point Amplitudes

(i) Mostly-plus three-gluon amplitude

\[ \mathcal{M}( -, -, + ) = \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} = \frac{\omega_1 \omega_2}{\omega_3} \frac{z_{12}^3}{z_{13} z_{23}} \]

\[ \tilde{\mathcal{A}}( -, -, + ) = 4 z_{21}^{1-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2-1} \delta(\bar{z}_{13}) \delta(\bar{z}_{23}) \int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)-1} d\omega_3 \]
Three-point Amplitudes

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logarithmically divergent in the infra-red and in ultra-violet
any cutoff would violate SL(2,C) symmetry

\[ = 2\pi \delta(\lambda_1 + \lambda_2 + \lambda_3) \]
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\]

logarithmically divergent in the infra-red and in ultra-violet
any cutoff would violate SL(2,C) symmetry

conformal transformation properties, read off:

\[
\begin{aligned}
  h_1 & = \frac{i}{2} \lambda_1, & \quad \bar{h}_1 & = 1 + \frac{i}{2} \lambda_1, \\
  h_2 & = \frac{i}{2} \lambda_2, & \quad \bar{h}_2 & = 1 + \frac{i}{2} \lambda_2, \\
  h_3 & = 1 + \frac{i}{2} \lambda_3, & \quad \bar{h}_3 & = \frac{i}{2} \lambda_3,
\end{aligned}
\]

\[
\Delta_n = 1 + i\lambda_n
\]

\[
J_1 = J_2 = -1, \quad J_3 = +1
\]

Pasterski, Shao, Strominger, 2017
(ii) Mostly-plus three-graviton amplitude

\[ \mathcal{M}(---,---,+++) = \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2} = \frac{\omega_1^2 \omega_2^2}{\omega_3^2} \frac{z_{12}^6}{z_{13}^2 z_{23}^2} \]

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\[ \mathcal{M}(- - , - - , + + ) = \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2} = \frac{\omega_1^2 \omega_2^2}{\omega_3^2} \frac{z_{12}^6}{z_{13}^2 z_{23}^2} \]

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The degree of this divergence will grow with the number of external gravitons, reflecting the violation of unitarity bounds at each order of perturbative Einstein's gravity
(ii) Mostly-plus three-graviton amplitude

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The degree of this divergence will grow with the number of external gravitons, reflecting the violation of unitarity bounds at each order of perturbative Einstein's gravity.

Conformal transformation properties, read off:

\[
\begin{align*}
h_1 &= -\frac{1}{2} + \frac{i}{2} \lambda_1, \\
h_2 &= -\frac{1}{2} + \frac{i}{2} \lambda_2, \\
h_3 &= \frac{3}{2} + \frac{i}{2} \lambda_3, \\
\bar{h}_1 &= \frac{3}{2} + \frac{i}{2} \lambda_1, \\
\bar{h}_2 &= \frac{3}{2} + \frac{i}{2} \lambda_2, \\
\bar{h}_3 &= -\frac{1}{2} + \frac{i}{2} \lambda_3
\end{align*}
\]

\[
\Delta_n = 1 + i\lambda_n
\]

\[
\begin{align*}
J_1 &= J_2 = -2, \\
J_3 &= +2
\end{align*}
\]
Four-point Gauge Amplitudes

\[ r = \frac{z_{12} z_{34}}{z_{23} z_{41}} \]

conformal invariant
cross-ratio on \( CS^2 \)

actually:

\[
\frac{s_{23}}{s_{12}} = \frac{1}{r} = -\frac{u}{s} = \sin^2\left(\frac{\theta}{2}\right)
\]

\[ s = s_{12} = (p_1 + p_2)^2 \]

\[ u = -s_{23} = (p_2 - p_3)^2 \]

\( \theta \) = scattering angle in center of mass frame
Four-point Gauge Amplitudes

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\[ s = s_{12} = (p_1 + p_2)^2 \]

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\[ \theta = \text{scattering angle in center of mass frame} \]

\[ \tilde{A}(-,-,+,+) = 8\pi \delta(r - \bar{r}) \delta\left(\sum_{n=1}^{4} \lambda_n\right) \]

\[ \times \left( \prod_{i < j}^{4} z_{ij}^{\frac{5}{3}} - h_i - h_j - \bar{z}_{ij}^{\frac{5}{3}} - \bar{h}_i - \bar{h}_j \right) r^{\frac{5}{3}} (r - 1)^{\frac{2}{3}} \theta(r - 1) \]
type I superstring theory:

\[ \tilde{A}_I(-, -, +, +) = 4 \left( \alpha' \right)^\beta \delta(r - \vec{r}) \theta(r - 1) \left( \prod_{i < j}^4 \frac{h_{ij}}{z_{ij}^3} - h_i - h_j \frac{\bar{h}_i - \bar{h}_j}{z_{ij}^3} \right) \]

\[ \times r^{\frac{5-\beta}{3}} (r - 1)^{\frac{2-\beta}{3}} I(r, \beta) \]
type I superstring theory:

\[
\tilde{A}_I(-, -, +, +) = 4 (\alpha')^\beta \delta(r - \bar{r}) \theta(r - 1) \left( \prod_{i < j} \frac{n_i}{z_{i,j}^3 - h_i - h_j} \frac{n_i}{z_{i,j}^3 - \bar{h}_i - \bar{h}_j} \right) \\
\times r^{\frac{5-\beta}{3}} (r - 1)^{\frac{2-\beta}{3}} I(r, \beta)
\]

\[
\beta := -\frac{i}{2} \sum_{n=1}^{4} \lambda_n
\]

\[
I(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2} \int_0^1 dx \frac{1}{x} [r \ln x - \ln(1 - x)]^{\beta - 1}
\]

\[
I(r, \beta) = 2\pi \delta \left( \sum_{n=1}^{4} \lambda_n \right) \\
+ \frac{i\pi}{2} (-r)^{\beta - 1} \sinh \left( \frac{1}{2} \sum_{n=1}^{4} \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} \zeta \left( -\frac{i}{2} \sum_{n=1}^{4} \lambda_n - k, \{1\}^k \right)
\]
Remarks:

• no $\alpha'$ - expansion (trivial dependence on $\alpha'$)!
• instead expansion in small scattering angle

$$r^{-1} = \sin^2 \left( \frac{\theta}{2} \right)$$

• all heavy string modes participate on same footing

• field-theory is recovered in the limit of forward scattering $\theta = 0$
Remarks:

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• all heavy string modes participate on same footing
• field-theory is recovered in the limit of forward scattering $\theta = 0$

Question: celestial CFT$_2$ \begin{align*}
\text{string world-sheet CFT}_2
\end{align*}
any relation?
String world-sheet as celestial sphere

\[ \mathcal{M}_I(\ -\ ,\ -\ ,\ +\ ,\ +\ ) = \mathcal{M}(\ -\ ,\ -\ ,\ +\ ,\ +\ ) F_I(s, u) \]

with string formfactor:

\[ F_I(s, u) = -\alpha' s_{12} B(-\alpha s_{12}, 1 + \alpha' s_{23}) = -s B(-s, 1 - u) = \frac{\Gamma(1 - s)\Gamma(1 - u)}{\Gamma(1 - s - u)} \]
String world-sheet as celestial sphere

\[ \mathcal{M}_I(-, - , +, +) = \mathcal{M}(-, - , +, +) F_I(s, u) \]

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consider high-energy limit:

\[ B(-s, 1 - u) = \int_0^1 x^{-1-s}(1 - x)^{a s} \]

saddle-point approximation:

\[ x_0 = \frac{1}{1 - a} \in CS^2 \quad a = r^{-1} < 0 \]
String world-sheet as celestial sphere

\[ \mathcal{M}_I(-, -, +, +) = \mathcal{M}(-, -, +, +) F_I(s, u) \]

with string formfactor:

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consider high-energy limit:

\[ \mathcal{B}(-s, 1 - u) = \int_0^1 x^{-1-s}(1 - x)^{as} \]

saddle-point approximation:

\[ x_0 = \frac{1}{1 - a} \in CS^2 \quad a = r^{-1} < 0 \]

world-sheet vertex position = point on celestial sphere

= solutions to scattering equations
String world-sheet as celestial sphere

$$\mathcal{M}_I(-, -, +, +) = \mathcal{M}(-, -, +, +) F_I(s, u)$$

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consider high-energy limit: 

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world-sheet vertex position = point on celestial sphere 
= solutions to scattering equations

celestial sphere = world-sheet

CFT on celestial sphere related to free world-sheet CFT
Four-point Gravity Amplitudes

heterotic graviton amplitude:

$$\tilde{A}_H(- -, --, ++, ++) = 4 (\alpha')^\beta - 1 \delta(r - \bar{r}) \theta(r - 1) \left( \prod_{i < j}^{4} \frac{h_i - h_j}{z_{ij}^{3/2}} \frac{\bar{h}_i - \bar{h}_j}{\bar{z}_{ij}^{3/2}} \right)$$

$$\times r^{11 - \beta} (r - 1)^{-1 - \beta/3} G(r, \beta)$$
Four-point Gravity Amplitudes

heterotic graviton amplitude:

\[ \tilde{A}_H(-, -, ++, +) = 4 \left( \alpha' \right)^{\beta-1} \delta(r - \bar{r}) \theta(r - 1) \left( \prod_{i < j} \frac{z_{ij}^3 - h_i - h_j}{\bar{z}_{ij}^3 - \bar{h}_i - \bar{h}_j} \right) \times r^{\frac{11-\beta}{3}} (r - 1)^{-\frac{1-\beta}{3}} G(r, \beta) \]

\[ G(r, \beta) = H(r, \beta - 1) \]

\[ H(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2\pi} \int_C \frac{d^2z}{|z|^2(1 - z)} \left[ r \ln |z|^2 - \ln |1 - z|^2 \right]^{\beta-1} \]

\[ H(r, \beta) = 2\pi \delta \left( \sum_{n=1}^{4} \lambda_n \right) + \frac{i\pi}{2} (-r)^{\beta-1} \sinh \left( \frac{1}{2} \sum_{n=1}^{4} \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} S^c \left( -\frac{i}{2} \sum_{n=1}^{4} \lambda_n - k - 1, k + 1 \right) \]
Properties:

• Finite result for any $r$!

  ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals

• UV completion provided by string theory

Alert:

• Divergent for $r \rightarrow \infty$ (field-theory limit)

every order in the perturbative expansion of gravity violates the unitarity bounds by growing powers of energy.

This uncontrollable growth at large energies poses an obstacle for transforming gravitational amplitudes to celestial sphere
Concluding remarks

- explicit and compact expressions for string amplitudes on celestial sphere

- string amplitudes on celestial sphere: no $\alpha'$ - expansion (trivial dependence on $\alpha'$)

- all heavy string modes participate on same footing

- high-energy limit: string world-sheet = celestial sphere

- gravity is UV completed: ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals
Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections?

for YM scattering amplitudes soft gluon theorem can be phrased in terms of tree-level Ward identities of $D=2$ Kac-Moody symmetry $J(z)$

He, Mitra, Strominger, arXiv:1503.02663

for quantum gravity scattering amplitudes the Lorentz symmetry is enhanced to infinite-dimensional local $D=2$ conformal symmetry $T(z)$ (full Virasoro symmetry)

Kapec, Lysov, Pasterski, Strominger, arXiv:1406.3312
Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections?

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understanding the nature of 2D CFT on celestial sphere would enable a holographic description of flat spacetime
construct complete set of on-shell wave functions in D=4: solves D=4 wave equations transforms as SL(2,Z) conformal primaries

<table>
<thead>
<tr>
<th>bases</th>
<th>momentum basis</th>
<th>conformal basis</th>
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<tbody>
<tr>
<td>notations</td>
<td>exp ($\pm ip \cdot x$)</td>
<td>$\varphi_{\Delta}^\pm (x^\mu; z, \bar{z}) = [-q(z \cdot x \mp i\epsilon)]^{-\Delta}$</td>
</tr>
<tr>
<td>labels</td>
<td>$p^\mu$ ($p^2 = 0$, $p^0 &gt; 0$)</td>
<td>$\Delta \in 1 + i\lambda$, $\lambda \in \mathbb{R}$, $z \in CS^2$</td>
</tr>
</tbody>
</table>

In the massless case the change of basis is furnished by **Mellin transform** of plane wave (or plus a shadow transform):

$$\varphi_{\Delta}^\pm (x^\mu; z, \bar{z}) = \int_0^\infty \omega^{\Delta-1} e^{\pm i\omega q \cdot x - \epsilon \omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{[-x \cdot q(z, \bar{z}) \mp i\epsilon]^{\Delta}}$$

$D=4$ scalar wave function (solution to Klein-Gordon equation), specified by $x$ and conformal dimension $\Delta = 1 + i\lambda$, $\lambda \in \mathbb{R}$

no dependence on $D=4$ momentum $p^\mu$
similarly for higher spin partners, e.g. spin 1:

\[
A_{\mu a}^{\Delta, \pm}(x^\mu; z, \bar{z}) = \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta} + \frac{\partial_a q \cdot x}{(-q \cdot x \mp i\epsilon)^{\Delta+1}} q_\mu
\]

convenient gauge representative:

\[
A_{\mu a}^{\Delta, \pm}(x^\mu; z, \bar{z}) = (\mp i)^\Delta \Gamma(\Delta) \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta}
\]

from Mellin transform:

\[
A_{\mu a}^{\Delta, \pm}(x^\mu; z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} \partial_a q_\mu e^{\pm i\omega q \cdot x - \epsilon\omega}
\]

Pasterski, Shao arXiv:1705.01027
Four-point Closed String Amplitudes

heterotic gauge amplitude:

\[
\tilde{A}_H(-,-,+,+) = 4(\alpha')^\beta \delta(r - \bar{r}) \theta(r - 1) \left( \prod_{i<j}^{4} \frac{h_i - h_j - \bar{h}_i + \bar{h}_j}{z_{ij}^3 - \bar{z}_{ij}^3} \right) \\
\times r^{\frac{5-\beta}{3}} (r - 1)^{\frac{2-\beta}{3}} H(r, \beta)
\]

\[
H(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2\pi} \int_C \frac{d^2z}{|z|^2 (1 - z)} \left[ r \ln |z|^2 - \ln |1 - z|^2 \right]^\beta - 1
\]

\[
H(r, \beta) = 2\pi \delta\left( \sum_{n=1}^{4} \lambda_n \right) \\
+ \frac{i\pi}{2} (-r)^{\beta - 1} \sinh \left( \frac{1}{2} \sum_{n=1}^{4} \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} S^c \left( -\frac{i}{2} \sum_{n=1}^{4} \lambda_n - k + 1, k + 1 \right)
\]
Single-valued Nielsen polylogarithms

Nielsen's polylogarithm functions (real):

\[ S_{n,p}(t) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_{0}^{1} \frac{dx}{x} \ln^{n-1} x \ln^p (1 - xt), \quad t \in \mathbb{C} \]

in particular:

\[ S_{n,p}(1) = \zeta(n + 1, \{1\}^{p-1}) \]

\[ \zeta(n + 1, \{1\}^{p-1}) = \zeta(n + 1, 1, \ldots, 1) = \sum_{n_1 > n_2 > \ldots > n_p} \frac{1}{n_1^{n_1+1} n_2 \ldots n_p} \]

Single-valued descendants:

\[ S^c(n, p) = \pi^{-1} \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_{\mathbb{C}} \frac{d^2 z}{|z|^2} (1 - z)^{-1} \ln^{n-1} |z|^2 \ln^p |1 - z|^2 \]

\[ S^c(n, p) = \text{sv} \ S_{n,p}(1) = \text{sv} \ \zeta(n + 1, \{1\}^{p-1}) \]