Strings on Celestial Sphere

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based on:

St.St., T.R. Taylor:

Strings on Celestial Sphere

arXiv:1806.05688

+ work to appear

Recap: from study of symmetries of scattering amplitudes:

deep connections between

gravity and gauge interactions

e.g.: KLT, BCJ, EYM (double-copy-construction)

(in momentum or twistor space)

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$$p_k^\mu, \quad k = 1, \ldots, N$$
$$p_k^2 = -m_k^2$$

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- with manifest translation symmetry
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D=4 Minkowski probably not the right space
to see **all** symmetries
of scattering amplitudes

Lorentz group in $\mathbf{R}^{1,D+1}$ is identical
to Euklidian D-dimensional conformal group SO(1,D+1)

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Scattering amplitudes in $\mathbf{R}^{1,D+1}$
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D=2: celestial sphere

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D=2: celestial sphere

Can 2D CFT on celestial sphere offer some new insight into
gauge-gravity connections ?

N particles on celestial sphere

$p_k \longrightarrow (E_k, z_k, \bar{z}_k)$ represent points z_k on CS^2

with: $$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3} \quad , \quad E_k = p_k^0 \ , \quad (\overrightarrow{p}_k)^2 = (p_k^0)^2$$

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$$p_k^\mu = E_k \left(1 \, , \frac{z_k + \bar{z}_k}{1 + |z_k|^2} \, , \frac{-i(z_k - \bar{z}_k)}{1 + |z_k|^2} \, , \frac{1 - |z_k|^2}{1 + |z_k|^2} \right)$$

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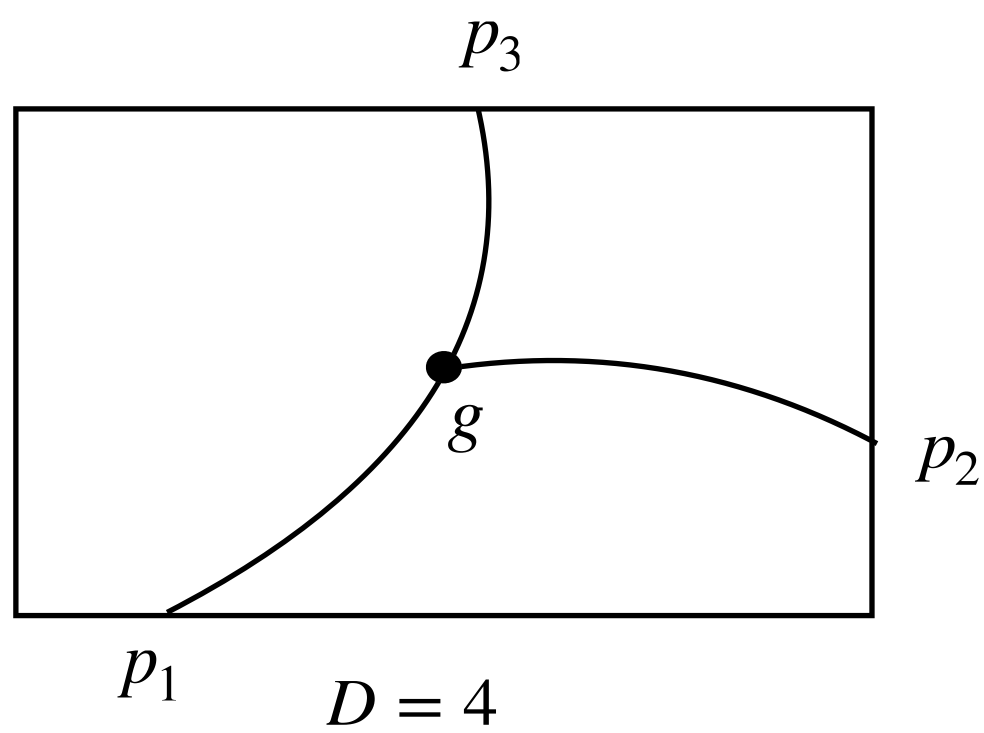
Lorentz symmetry: $$z \rightarrow \frac{az + b}{cz + d}$$ global conformal symmetry on CS^2

Amplitudes = conformal correlators of primary fields on CS^2



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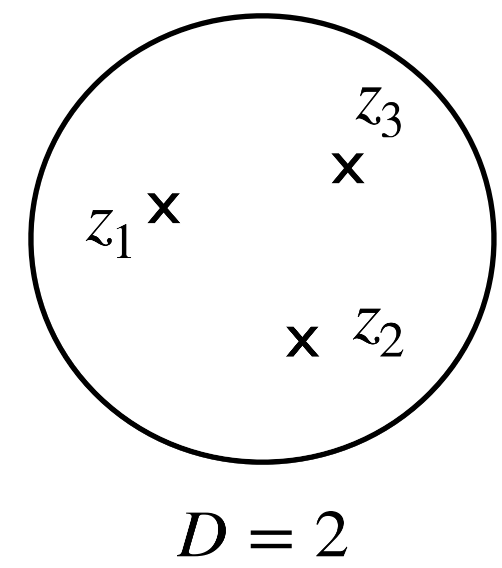
=



$$\sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3} \, |z_2 - z_3|^{h_2+h_3-h_1} \, |z_1 - z_3|^{h_1+h_3-h_2}}$$

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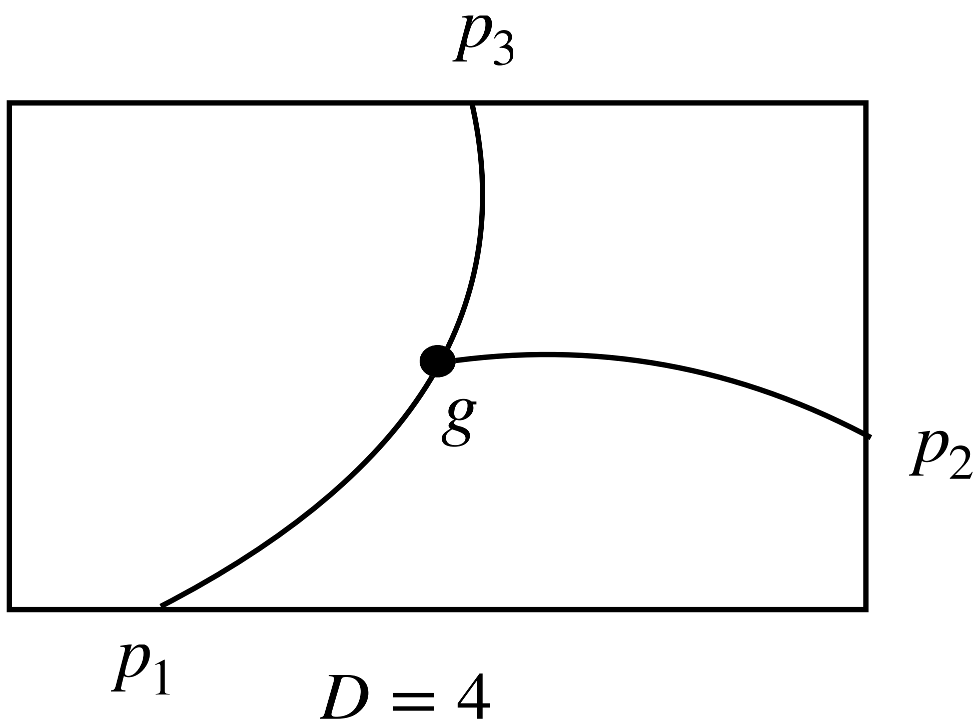

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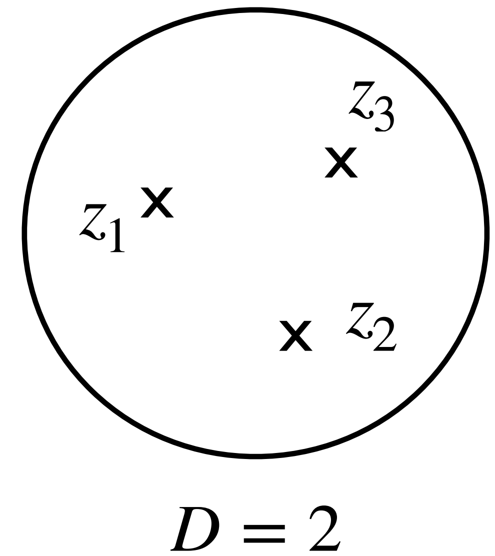

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→ D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

D=2 CFT correlators involve conformal wave packets

In practice: in momentum basis: plane waves with momentum p

conformal basis: conformal primary wave functions Δ

→ Mellin transformation

$$\tilde{\phi}(\Delta) = \int_0^{\infty} d\omega\, \omega^{\Delta-1}\, \phi(\omega)$$

In the massless case, with or without spin, the transition from momentum space to conformal primary wavefunctions with Δ_j is implemented by Mellin transform

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$$\mathscr{A}(\{p_i, \xi_j\}) = i(2\pi)^4\, \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{N} p_k\right)\, \mathscr{M}(\{p_i, \xi_j\})$$

Mellin transform, with: $\Delta_j = 1 + i\lambda_j$

$$\tilde{\mathscr{A}}_{\{\lambda_n\}}(z_n, \bar{z}_n) = \left(\prod_{n=1}^{N} \int_0^\infty \omega_n^{i\lambda_n}\, d\omega_n\right)\, \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k)$$

$$\times\, \mathscr{M}(\omega_n, z_n, \bar{z}_n)$$

Three-point Amplitudes

(i) Mostly-plus three-gluon amplitude

$$\mathcal{M}(-,-,+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} = \frac{\omega_1\omega_2}{\omega_3}\frac{z_{12}^3}{z_{13}z_{23}}$$

$$\tilde{\mathcal{A}}(-,-,+) = 4\, z_{21}^{1-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2-1}\, \delta(\bar{z}_{13})\delta(\bar{z}_{23}) \int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)-1} d\omega_3$$

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any cutoff would violate SL(2,C) symmetry

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conformal transformation properties, read off:

$$\left.\begin{aligned} h_1 &= \frac{i}{2}\lambda_1, & \bar{h}_1 &= 1+\frac{i}{2}\lambda_1, \\ h_2 &= \frac{i}{2}\lambda_2, & \bar{h}_2 &= 1+\frac{i}{2}\lambda_2, \\ h_3 &= 1+\frac{i}{2}\lambda_3, & \bar{h}_3 &= \frac{i}{2}\lambda_3, \end{aligned}\right\}$$

$$\Delta_n = 1 + i\lambda_n$$

$$J_1 = J_2 = -1, \quad J_3 = +1$$

Pasterski, Shao, Strominger, 2017

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Four-point Gauge Amplitudes

$$r = \frac{z_{12}\, z_{34}}{z_{23}\, z_{41}}$$

conformal invariant
cross-ratio on CS^2

actually: $$\frac{s_{23}}{s_{12}} = \frac{1}{r} = -\frac{u}{s} = \sin^2\left(\frac{\theta}{2}\right)$$

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$$\tilde{\mathcal{A}}(-,-,+,+) = 8\pi\, \delta(r - \bar{r})\, \delta\left(\sum_{n=1}^{4} \lambda_n\right) \times \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3} - h_i - h_j}\, \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j}\right) r^{\frac{5}{3}}\, (r-1)^{\frac{2}{3}}\, \theta(r-1)$$

Pasterski, Shao, Strominger, 2017

type I superstring theory:

$$\tilde{\mathcal{A}}_I(-,-,+,+) = 4\,(\alpha')^{\beta}\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$
$$\times\; r^{\frac{5-\beta}{3}}\,(r-1)^{\frac{2-\beta}{3}}\,I(r,\beta)$$

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$$\beta := -\frac{i}{2}\sum_{n=1}^{4}\lambda_n$$

$$I(r,\beta) = -\,\Gamma(1-\beta)\,\frac{r}{2}\int_0^1 \frac{dx}{x}\,\left[r\ln x - \ln(1-x)\right]^{\beta-1}$$

$$I(r,\beta) = 2\pi\,\delta\left(\sum_{n=1}^{4}\lambda_n\right) + \frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\zeta\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n - k,\{1\}^k\right)$$

Remarks:

- no α' - expansion (trivial dependence on α') !
- instead expansion in small scattering angle

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

- all heavy string modes participate on same footing
- field-theory is recovered in the limit of forward scattering $\theta = 0$

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Question: celestial CFT_2 / string world-sheet CFT_2 } → any relation ?

String world-sheet as celestial sphere

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with string formfactor:

$$F_I(s,u) = -\alpha' s_{12} B(-\alpha s_{12}, 1+\alpha' s_{23}) = -s\ B(-s, 1-u) = \frac{\Gamma(1-s)\Gamma(1-u)}{\Gamma(1-s-u)}$$

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$$B(-s,1-u) = \int_0^1 x^{-1-s}(1-x)^{as}$$

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$$\boxed{x_0 = \frac{1}{1-a} \in CS^2} \qquad a = r^{-1} < 0$$

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= solutions to scattering equations

→ celestial sphere = world-sheet → CFT on celestial sphere related to free world-sheet CFT

Four-point Gravity Amplitudes

heterotic graviton amplitude:

$$\tilde{\mathcal{A}}_H(--,--,++,++) = 4\ (\alpha')^{\beta-1}\ \delta(r-\bar{r})\ \theta(r-1)\ \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\ \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

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$$G(r,\beta) = H(r,\beta-1)$$

$$H(r,\beta) = -\,\Gamma(1-\beta)\,\frac{r}{2\pi}\int_{\mathbf{C}}\frac{d^2z}{|z|^2(1-z)}\,\left[r\ln|z|^2-\ln|1-z|^2\right]^{\beta-1}$$

$$H(r,\beta) = 2\pi\,\delta\left(\sum_{n=1}^{4}\lambda_n\right)$$
$$+\,\frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\;S^{\mathbf{c}}\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n-k-1,k+1\right)$$

Properties:

- Finite result for any r !

ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals

- UV completion provided by string theory

Alert:

- Divergent for $r \to \infty$ (field-theory limit)

every order in the perturbative expansion of gravity violates the unitarity bounds by growing powers of energy. This uncontrollable growth at large energies poses an obstacle for transforming gravitational amplitudes to celestial sphere

Concluding remarks

- explicit and compact expressions for string amplitudes on celestial sphere
- string amplitudes on celestial sphere: no α' - expansion (trivial dependence on α')
- all heavy string modes participate on same footing
- high-energy limit: string world-sheet = celestial sphere
- gravity is UV completed: ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals

Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections ?

for YM scattering amplitudes soft gluon theorem
can be phrased in terms of
tree-level Ward identities of D=2 Kac-Moody symmetry $J(z)$

He, Mitra, Strominger, arXiv:1503.02663

for quantum gravity scattering amplitudes
the Lorentz symmetry is enhanced
to infinite-dimensional local D=2 conformal symmetry $T(z)$
(full Virasoro symmetry)

Kapec, Lysov, Pasterski, Strominger, arXiv:1406.3312

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understanding the nature of 2D CFT on celestial sphere
would enable a **holographic description of flat spacetime**

construct complete set of on-shell wave functions in D=4:
solves D=4 wave equations
transforms as SL(2,Z) conformal primaries

	momentum basis	conformal basis
bases	plane waves	conformal primary wavefunctions
notations	$\exp(\pm ip \cdot x)$	$\varphi^{\pm}_{\Delta}(x^{\mu}; z, \bar{z}) = [-q(z \cdot x \mp i\epsilon]^{-\Delta}$
labels	$p^{\mu}\ (p^2 = 0,\ p^0 > 0)$	$\Delta \in 1 + i\lambda, \quad \lambda \in \mathbf{R},\ z \in CS^2$

in the massless case the change of basis is furnished
by **Mellin transform** of plane wave (or plus a shadow transform):

$$\varphi^{\pm}_{\Delta}(x^{\mu}; z, \bar{z}) = \int_0^{\infty} \omega^{\Delta-1} e^{\pm i\omega q\cdot x - \epsilon\omega} = \frac{(\mp i)^{\Delta}\Gamma(\Delta)}{[-x \cdot q(z, \bar{z}) \mp i\epsilon]^{\Delta}}$$

D=4 scalar wave function (solution to Klein-Gordon equation),

specified by x and conformal dimension $\Delta = 1 + i\lambda, \quad \lambda \in \mathbf{R}$

no dependence on D=4 momentum p^{μ}

similarly for higher spin partners, e.g. spin 1:

$$A_{\mu a}^{\Delta,\pm}(x^\mu; z, \bar{z}) = \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta} + \frac{\partial_a q \cdot x}{(-q \cdot x \mp i\epsilon)^{\Delta+1}} \; q_\mu$$

convenient gauge representative:

$$A_{\mu a}^{\Delta,\pm}(x^\mu; z, \bar{z}) = (\mp i)^\Delta \; \Gamma(\Delta) \, \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta}$$

from Mellin transform:

$$A_{\mu a}^{\Delta,\pm}(x^\mu; z, \bar{z}) = \int_0^\infty d\omega \; \omega^{\Delta-1} \; \partial_a q_\mu \; e^{\pm i\omega q\cdot x - \epsilon\omega}$$

Pasterski, Shao arXiv:1705.01027

Four-point Closed String Amplitudes

heterotic gauge amplitude:

$$\tilde{A}_H(-,-,+,+) = 4(\alpha')^\beta\ \delta(r-\bar{r})\ \theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\ \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right) \times\ r^{\frac{5-\beta}{3}}(r-1)^{\frac{2-\beta}{3}}\ H(r,\beta)$$

$$H(r,\beta) = -\,\Gamma(1-\beta)\,\frac{r}{2\pi}\int_{\mathbf{C}}\frac{d^2z}{|z|^2(1-z)}\left[r\ln|z|^2-\ln|1-z|^2\right]^{\beta-1}$$

$$H(r,\beta) = 2\pi\,\delta\left(\sum_{n=1}^{4}\lambda_n\right) + \frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\ S^{\mathbf{c}}\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n-k-1,k+1\right)$$

Single-valued Nielsen polylogarithms

Nielsen's polylogarithm functions (real):

$$S_{n,p}(t) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dx}{x} \ \ln^{n-1} x \ \ln^p(1-xt) \quad , \quad t \in \mathbf{C}$$

in particular:

$$\boxed{S_{n,p}(1) = \zeta(n+1, \{1\}^{p-1})}$$

$$\zeta(n+1,\{1\}^{p-1}) = \zeta(n+1, \underbrace{1,\ldots,1}_{p-1}) = \sum_{n_1>n_2>\ldots>n_p} \frac{1}{n_1^{n+1} n_2 \cdots n_p}$$

Single-valued descendants:

$$S^{\mathbf{c}}(n,p) = \pi^{-1} \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_{\mathbf{C}} \frac{d^2 z}{|z|^2} (1-z)^{-1} \ \ln^{n-1} |z|^2 \ \ln^p |1-z|^2$$

$$\boxed{S^{\mathbf{c}}(n,p) = \text{sv } S_{n,p}(1) = \text{sv } \zeta\left(n+1,\{1\}^{p-1}\right)}$$