Einstein Double Field Equations

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Introduction

General Relativity is a successful theory of gravity.

- Equivalence Principle: gravity = acceleration; at every spacetime point, $\exists$ local inertial frame in which laws of Physics are invariant.
- Mathematically: spacetime is a Riemannian manifold, endowed with a dynamical metric, $g_{\mu\nu}$, and associated covariant derivative

$$\nabla_\mu = \partial_\mu + \gamma_\mu + \omega_\mu \, ,$$

  where
  - Christoffel symbols $\gamma_\mu \Rightarrow$ diffeomorphism invariance,
  - spin connection $\omega_\mu \Rightarrow$ local Lorentz invariance.
- Geometry $\Leftrightarrow$ Matter; expressed via Einstein's equations

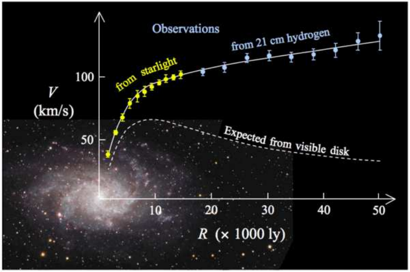
$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \, .$$

Motivation: “dark universe"

- GR accurately describes astrophysical/cosmological phenomena: perihelion precession, gravitational lensing, Big Bang cosmology...
- However, some results cannot be explained by GR + visible matter alone, eg. galaxy rotation curves.
- Kepler/Newton/GR: orbital velocity

$$V^2 = \frac{GM}{R} ,$$

does not match observations.



Broadly, two classes of solutions to such problems:

1. GR is correct, but there is additional dark matter, dark energy, . . .
2. Theory of gravity should be modified (see A.C. Davis’ talk).

Motivation: string theory

In GR, the metric $g_{\mu\nu}$ is the only gravitational field.
In string theory, the closed-string massless sector always includes:

- the metric, $g_{\mu\nu}$;
- an antisymmetric 2-form potential, $B_{\mu\nu}$;
- the dilaton, ϕ.

Furthermore, these fields transform into each other under T-duality.

Natural stringy extension of General Relativity:

Consider $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ as the fundamental gravitational multiplet.

This is the idea of Stringy Gravity, which can be realized using the mathematical formalism of Double Field Theory (DFT).

In Stringy Gravity, the additional degrees of freedom $B_{\mu\nu}$ and ϕ augment gravity beyond GR, allowing new types of solutions.

A brief introduction to Double Field Theory

- In Double Field Theory (Hull, Zwiebach; 2009) we describe D-dim. physics using $D+D$ coordinates, $x^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, \ldots, D+D$.
- $\exists$ an $\mathbf{O}(D,D)$ T-duality gauge symmetry; doubled vector indices are raised and lowered using the $\mathbf{O}(D,D)$-invariant metric: $$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
- $\exists$ doubled diffeomorphisms acting on vectors ξ^A, etc.
- $\exists$ twofold local Lorentz symmetry: $\mathbf{Spin}(1, D-1) \times \mathbf{Spin}(D-1, 1)$, with local metrics $\eta_{pq} = \mathrm{diag}(-++\cdots+)$, $\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$.
- Equivalence relation: $x^A \sim x^A + \Delta^A(x)$, for $\Delta^A \sim \partial^A = (\partial_\nu, \tilde{\partial}^\mu)$.
- This is equivalent to the section condition: $\partial_A \partial^A = 2\,\partial_\mu \tilde{\partial}^\mu = 0$.
- Natural choice: $\tilde{\partial}^\nu = 0$. Thus the D coordinates $\{\tilde{x}_\mu\}$ are gauged, and their gauge orbits correspond to points in the resulting D-dimensional spacetime which is spanned by $\{x^\nu\}$.

Field content of Double Field Theory

- The basic fields of Double Field Theory are: $\{d, \mathcal{H}_{AB}\}$, the DFT dilaton and the symmetric $\mathbf{O}(D, D)$ metric, respectively.
- After imposing the section condition, these reduce to the closed string massless sector, $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, e.g. $e^{-2d} \simeq e^{-2\phi}\sqrt{-g}$.
- Semi-covariant master derivative, $\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A$, where Γ_{ABC} and $\{\Phi_{Apq}, \bar{\Phi}_{A\bar{p}\bar{q}}\}$ are DFT Christoffel and spin connections.
- $\nexists$ normal coordinates where $\Gamma_{ABC} = 0 \Rightarrow$ no equivalence principle! ($B_{\mu\nu}$ sources string; EP does not apply to extended objects.)
- Use projectors, $P_{AB} = \frac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB})$, $\bar{P}_{AB} = \frac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})$, to construct the fully covariant DFT Ricci tensor $S_{p\bar{q}}$ and scalar $S_{(0)}$.
- Upon Riemannian backgrounds ($\tilde{\partial}^\mu = 0$), reduces to e.g.

$$S_{(0)} = R + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \; .$$

This gives the spacetime Lagrangian for Stringy Gravity.

Stringy energy-momentum tensor

We will now consider DFT as Stringy Gravity coupled to matter. The action is

$$\int_{\Sigma} e^{-2d}\left[\,\frac{1}{16\pi G}S_{(0)} + L_{\mathrm{matter}}(\Upsilon_a)\,\right]\,,$$

where the integral is performed over a D-dimensional section Σ.

Note: $\mathbf{O}(D,D)$ symmetry $\Rightarrow$ proper distance, geodesic motion, etc. have a natural covariant definition in string (Jordan) frame.

The resulting equations of motion are

$$S_{p\bar{q}} = 8\pi G K_{p\bar{q}}\,,\qquad S_{(0)} = 8\pi G T_{(0)}\,,\qquad \frac{\delta L_{\mathrm{matter}}}{\delta \Upsilon_a} \equiv 0\,.$$

Here the stringy energy-momentum tensor has components

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta \bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^{p}}\right)\,,\quad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\mathrm{matter}}\right)}{\delta d}\,,$$

where V and $\bar{V}$ are DFT vielbeins: $P_A{}^B = V_{Ap}V^{Bp}$; $\bar{P}_A{}^B = \bar{V}_{A\bar{p}}\bar{V}^{B\bar{p}}$.

Einstein Double Field Equations

Analogously to GR, we can define the stringy Einstein curvature tensor which is covariantly conserved,

$$G_{AB} = 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)}\,.$$

This implies that the energy-momentum tensor can be written similarly,

$$T_{AB} := 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \mathcal{D}_A T^{AB} \equiv 0 \qquad \text{(on-shell)}\,.$$

Hence the Einstein Double Field Equations can be summarized as

$$G_{AB} = 8\pi G T_{AB}\,.$$

Note: unlike in GR, the DFT Ricci tensor is traceless $\Rightarrow$ the $S_{(0)} \propto T_{(0)}$ part is an essential and independent component of the equations.

Riemannian backgrounds

Riemannian backgrounds, section condition $\tilde{\partial}^{\mu} = 0$:

- Einstein Double Field Equations reduce to the usual closed-string equations, plus source terms from $K_{\mu\nu} = 2e_{\mu}{}^{p}\bar{e}_{\nu}{}^{q}K_{p\bar{q}}$ and $T_{(0)}$:

$$\begin{aligned} R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} &= 8\pi G K_{(\mu\nu)} \; ; \\ \bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) &= 16\pi G e^{-2\phi}K_{[\mu\nu]} \; ; \\ R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 8\pi G T_{(0)} \; . \end{aligned}$$

- Asymmetric $K_{\mu\nu}$ possible (e.g. fermions, strings) $\rightarrow$ source for H.
- In addition, the conservation laws reduce to

$$\begin{aligned} \nabla^{\mu}K_{(\mu\nu)} - 2\partial^{\mu}\phi\, K_{(\mu\nu)} + \tfrac{1}{2}H_{\nu}{}^{\lambda\mu}K_{[\lambda\mu]} - \tfrac{1}{2}\partial_{\nu}T_{(0)} &\equiv 0 \; , \\ \nabla^{\mu}\left(e^{-2\phi}K_{[\mu\nu]}\right) &\equiv 0 \; , \end{aligned}$$

which are automatically satisfied on-shell.

Spherically symmetric solution

Finally, we wish to look for solutions, e.g. spherically symmetric, asymptotically flat, static, Riemannian, regular solutions in $D = 4$.

- Spherical, static ansatz for $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$:

$$\mathrm{d}s^2 = e^{2\phi(r)}\left[-A(r)\mathrm{d}t^2 + A^{-1}(r)\mathrm{d}r^2 + A^{-1}(r)C(r)\,\mathrm{d}\Omega^2\right];$$
$$B_{(2)} = B(r)\cos\vartheta\,\mathrm{d}r\wedge\mathrm{d}\varphi + h\cos\vartheta\,\mathrm{d}t\wedge\mathrm{d}\varphi\,; \quad \phi = \phi(r)\,.$$

- Note: physical distance given by "areal radius", $R := e^{\phi}\sqrt{C/A}$.
- Consider solutions with matter only up to a cutoff radius, r_c. Solution for $r \geq r_\mathrm{c}$ is the known DFT vacuum solution, which has four free parameters, $\{\alpha, a, b, h\}$ (Ko, Park, Suh; 2017).
- Boundary cond's: flat spacetime at $r \to \infty$; regular sol'n at $r = 0$.
- New feature: matching $r < r_\mathrm{c}$ and $r \geq r_\mathrm{c}$ regions gives $\{\alpha, a, b, h\}$ in terms of the interior energy-momentum tensor.
- Note: $B(r) = 0$, i.e. only "electric" H-flux, $h = H_{t\vartheta\varphi}$, for $r \geq r_\mathrm{c}$.
- At $r = \alpha$, $A = C = R = 0$ and $\phi \to \pm\infty \Rightarrow$ require $r_\mathrm{c} > \alpha$.

Effective mass

- Define an effective mass $M(r)$ via the centripetal acceleration measured in Newtonian gravity with the areal radius R. From the radial geodesic equation,

$$\frac{GM(r)}{R^2} \equiv R\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 .$$

- When $r \to \infty$,

$$\begin{aligned} M_\infty \equiv \lim_{r\to\infty} M(r) &= \frac{a + b\sqrt{1 - h^2/b^2}}{2G} \\ &= \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, e^{-2d}\left(\frac{1}{8\pi G}\left|H_{t\vartheta\varphi}H^{t\vartheta\varphi}\right| - 2K_t{}^t\right) . \end{aligned}$$

- $M_\infty \geq 0 \Leftrightarrow$ “weak energy condition" of Stringy Gravity satisfied.
- Note that b can take either sign $\Rightarrow b < 0$ may be problematic.

Small-radius behavior

- The effective mass $M(r)$ appears in the expansion of g_{tt} and corresponds to the gravitational "force".
- In particular, if $M(r) = 0$ for some finite $r = r_M \geq r_\mathrm{c}$, then for $r < r_M$ the gravitational force may become repulsive!
- On the other hand, $M(r) \propto \Omega(r)^{-1}$, where the function

$$\Omega(r) := r - \alpha + \tfrac{1}{2}\sqrt{a^2 + b^2} - \tfrac{1}{2}a \pm \tfrac{1}{2}|b|\sqrt{1 - e^{-4\phi}\, h^2/b^2} \sim \frac{\mathrm{d}R}{\mathrm{d}r}\,.$$

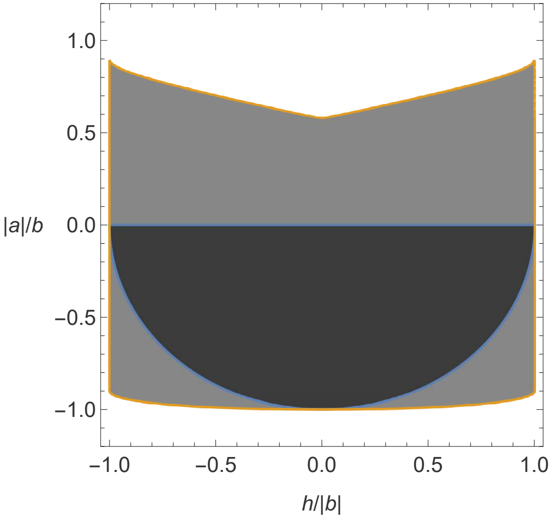
  For sufficiently small $r > \alpha$, the sign of the last term is always negative $\Rightarrow \exists\ r_\Omega > \alpha$ such that $\Omega(r_\Omega) = 0$ and $M(r)$ diverges.

The small-r (and thus small-R) behavior depends on the relative size of r_M and r_Ω, which in turn depends on the parameters $\{a, b, h\}$:

1. $r_M > r_\Omega \Rightarrow$ gravity repulsive for $r_\Omega \leq r < r_M$, repulsive wall at $r = r_\Omega$;
2. $r_M \leq r_\Omega$ or $\nexists \Rightarrow M(r) \geq 0$ for $r > r_\Omega$, attractive singularity at $r = r_\Omega$.

Constraints on parameters

- Parameter space for $h/|b|$ and $|a|/b$.
- In the gray region $r_M > r_\Omega$, whereas this is not satisfied in the outer white region.
- The black region corresponds to violation of the weak energy condition ($M_\infty < 0$) $\Rightarrow$ excluded.
- For physical solutions, require $r_c > r_\Omega$.



Summary

- Stringy Gravity considers the closed string massless sector $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ to be the fundamental gravitational multiplet $\Rightarrow (D^2+1)$ degrees of freedom, thus richer spectrum.
- We studied Double Field Theory as Stringy Gravity in the presence of matter. Imposing on-shell energy-momentum conservation gives the Einstein Double Field Equations,

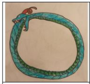
$$G_{AB} = 8\pi G T_{AB} \, .$$

- For spherically symmetric regular solutions in $D=4$, at small radii (Note: small R/GM_∞) the gravitational force is modified and can become repulsive. Applications to modified gravity?

Further investigation: non-Riemannian spacetimes; string cosmology & thermodynamics (R. Brandenberger's talk); tests against observations.

Uroboros spectrum

- In Stringy Gravity, the additional degrees of freedom $B_{\mu\nu}$ and ϕ augment gravity beyond GR, allowing new types of solutions.
- E.g. $D = 4$, spherical, static case: Stringy Gravity has 4 free parameters (c.f. 1 parameter in GR, the Schwarzschild mass).
- Gravity is modified at "short" distances (Ko, Park, Suh; 2017); best expressed in terms of the dimensionless variable R/MG.
- Anomalous behavior of large astrophysical objects corresponds to this parameter range, as very large $M \Rightarrow$ small $R/MG \lesssim 10^7$.

	Electron ($R \simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_{\odot}G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M \propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^{5}$	0^+

'Uroboros' spectrum of the dimensionless Radial variable normalized by Mass in natural units.
The orbital speed of rotation curves is also dimensionless, and depends on the single variable, $R/(MG)$.